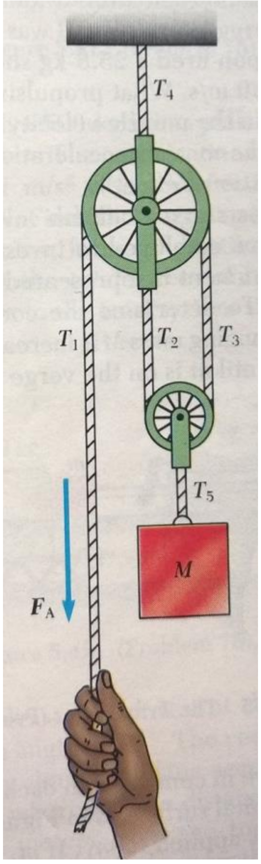
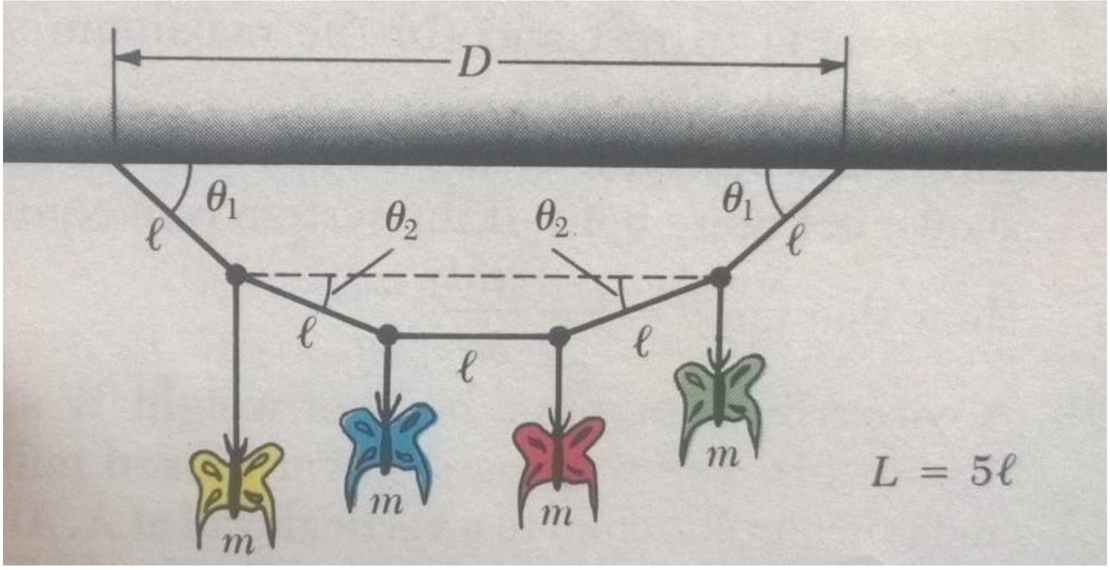
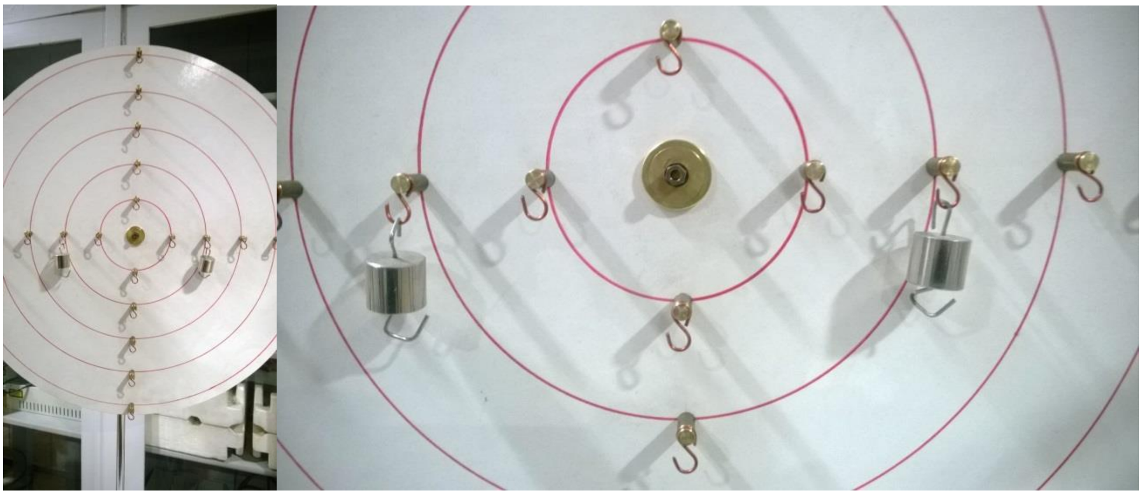


T_4
T_1
T_2
T_3
T_5
M
F_A

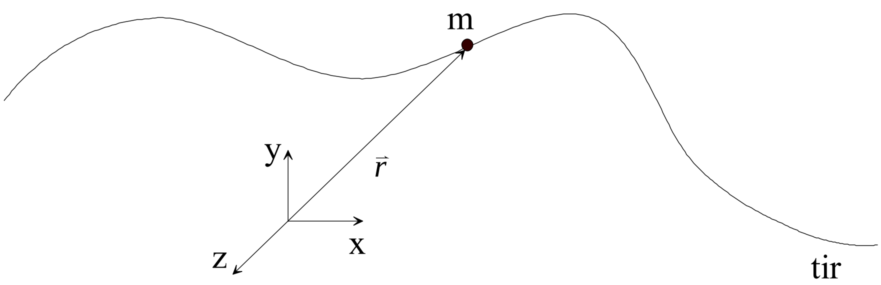

D
θ1
θ2
θ2
θ1
ℓ
ℓ
ℓ
ℓ
ℓ
m
m
m
m
L = 5ℓ



Izrek o vrtilni količini

Točkasto telo

$$\vec{r}(t) = (x,\ y,\ z)$$



hitrost $\vec{v} = \dfrac{d\vec{r}}{dt}$ pospešek $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt}$

II. Newton-ov zakon za gibanje točkastega telesa:

$$m\vec{a} = \vec{F}$$

$$\vec{r} \times m\vec{a} = \vec{r} \times \vec{F} = \vec{M}$$

$\vec{M}$ = **rezultantni navor vseh zunanjih sil**

$$\frac{d}{dt}(\vec{r} \times m\vec{v}) = \frac{d\vec{r}}{dt} \times m\vec{v} + \vec{r} \times m\frac{d\vec{v}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times m\vec{a} = \vec{r} \times m\vec{a}$$

$$\frac{d(\vec{r} \times m\vec{v})}{dt} = \vec{M}$$

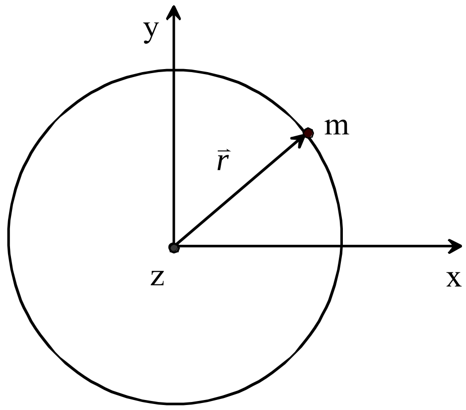
vrtilna količina točkastega telesa: $\vec{\Gamma} = \vec{r} \times m\vec{v}$

$$\boxed{\frac{d\vec{\Gamma}}{dt} = \vec{M}}$$

$$\mathrm{d}\vec{\Gamma} = \vec{M}\mathrm{d}t$$

$$\int_{t_1}^{t_2} \vec{M}\,\mathrm{d}t = \vec{\Gamma}_2 - \vec{\Gamma}_1 = \Delta\vec{\Gamma}$$

Primer: enakomerno **krožečo točkasto maso**



$$\vec{\Gamma} = \vec{r} \times m\vec{v} \qquad\qquad v = \omega r$$

$$\Gamma = r\,mv = r\,m\omega r = mr^2\omega \qquad \vec{\Gamma} = J\,\vec{\omega}$$

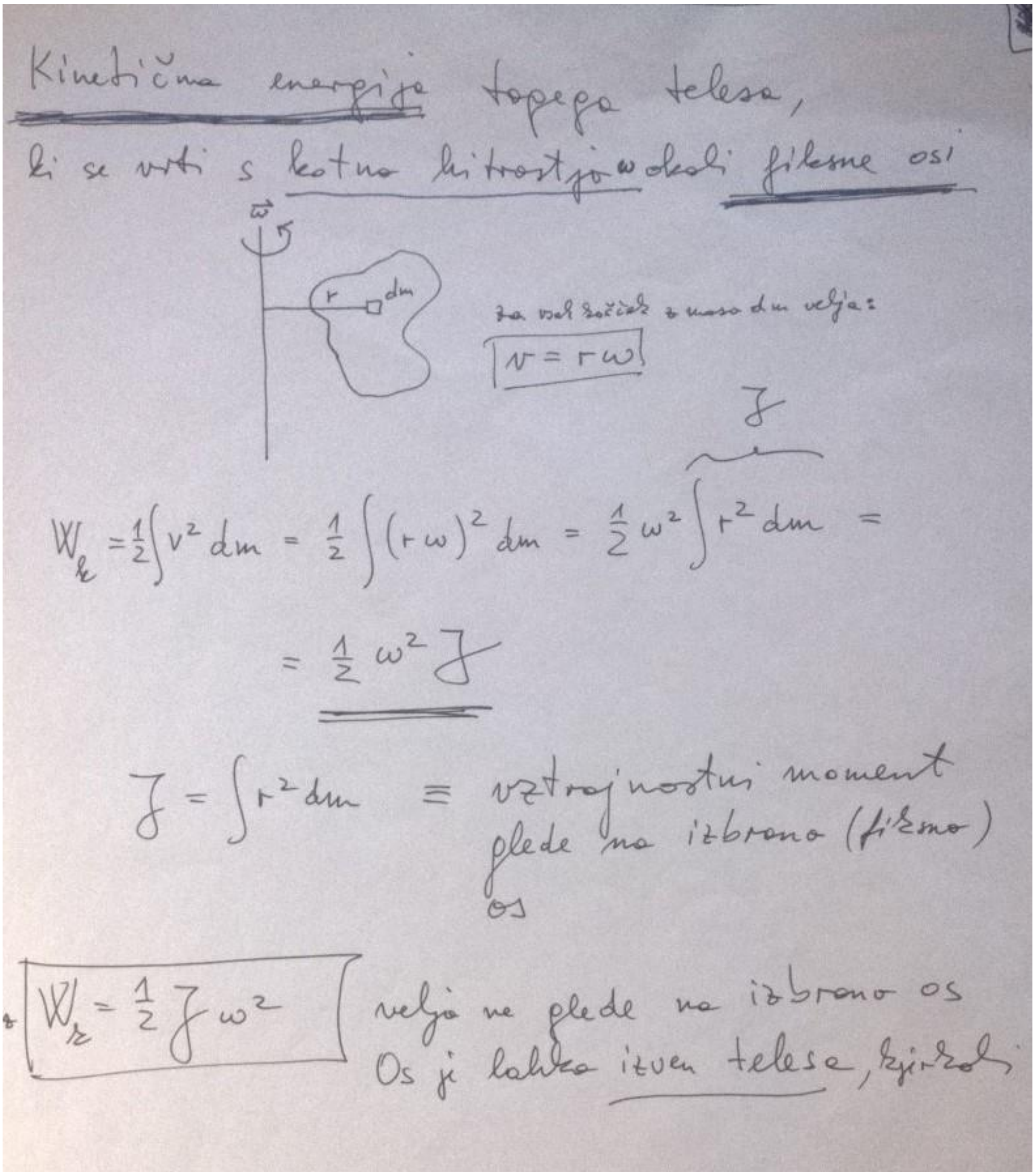
$$W_k = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\,\omega^2 = \frac{1}{2}J\,\omega^2$$

kjer definiramo **vztrajnostni moment krožeče točkaste mase**

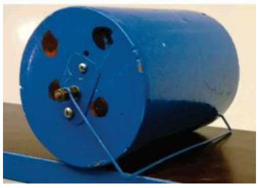
$$J = mr^2$$

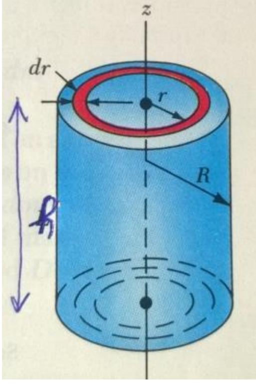
$$\vec{M} = \frac{\mathrm{d}\vec{\Gamma}}{\mathrm{d}t} = J\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} = J\,\vec{\alpha} \qquad \vec{\alpha} = \frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}$$

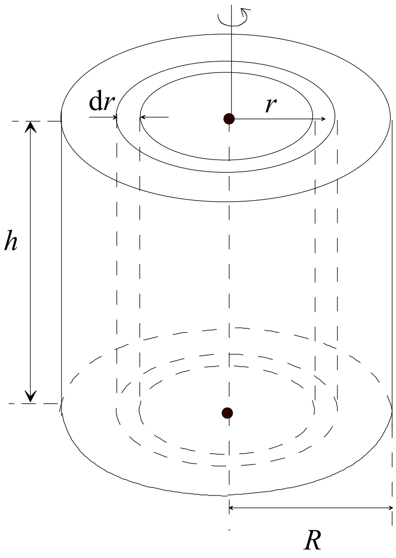
$$\int_{t_1}^{t_2} \vec{M}\,\mathrm{d}t = J\vec{\omega}_2 - J\vec{\omega}_1$$



Zgled: vztrajnostni moment **homogenega valja okrog geometrijske osi**







R ≡ polmer valja
h ≡ višina valja

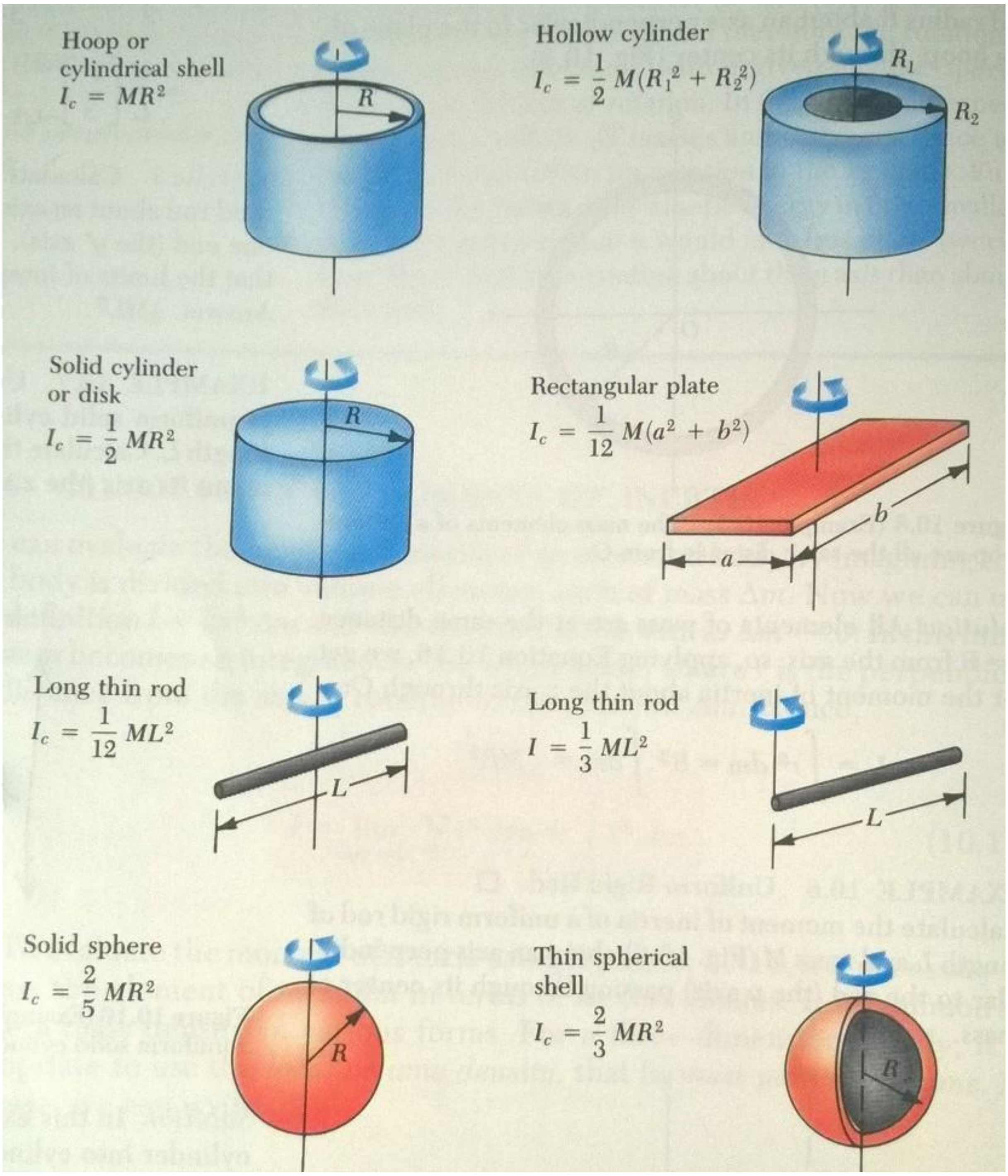
$$J = \int r^2 \,\mathrm{d}m = \int r^2 \rho \,\mathrm{d}V$$

ρ = gostota valja

$$\mathrm{d}V = 2\pi\, r \,\mathrm{d}r\, h$$

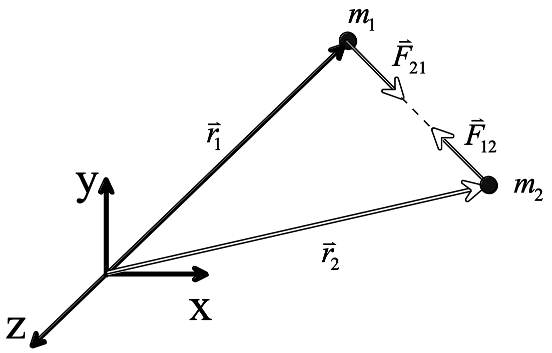
$$J = \int_0^R r^2 \rho\, 2\pi\, r\, h\,\mathrm{d}r = 2\pi\, h\rho \int_0^R r^3 \mathrm{d}r = 2\pi\, h\rho \left.\frac{r^4}{4}\right|_0^R = \pi R^4 \rho \frac{h}{2} = \frac{1}{2} m\, R^2$$

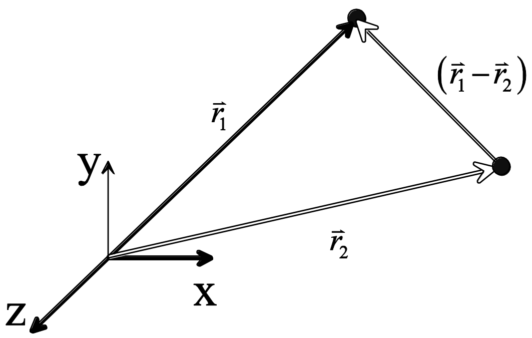
celotna masa valja $m = \pi\, R^2\, h\, \rho$


Hoop or cylindrical shell
$I_c = MR^2$
R
Hollow cylinder
$I_c = \frac{1}{2} M(R_1^2 + R_2^2)$
R_1
R_2
Solid cylinder or disk
$I_c = \frac{1}{2} MR^2$
R
Rectangular plate
$I_c = \frac{1}{12} M(a^2 + b^2)$
b
a
Long thin rod
$I_c = \frac{1}{12} ML^2$
L
Long thin rod
$I = \frac{1}{3} ML^2$
L
Solid sphere
$I_c = \frac{2}{5} MR^2$
R
Thin spherical shell
$I_c = \frac{2}{3} MR^2$
R

Sistem točkastih mas

Poseben primer **: sistem dveh točkastih delcev** III. Newtonov zakon : $\vec{F}_{12}+\vec{F}_{21}=\vec{0}$





$$\sum_i \vec{M}_i=\vec{r}_1\times\vec{F}_{21}+\vec{r}_2\times\vec{F}_{12}=\vec{r}_1\times\vec{F}_{21}+\vec{r}_2\times\left(-\vec{F}_{21}\right)$$

$$\sum_i \vec{M}_i=\left(\vec{r}_1-\vec{r}_2\right)\times\vec{F}_{21}=\vec{0}$$

POSPLOŠITEV: **navor vseh notranjih sil nič**, če so dvodelčne notranje sile **centralne**

vrtilna količina množice točkastih mas :

$$\vec{\Gamma}=\sum_i \vec{r}_i\times m_i\vec{v}_i$$

$$\frac{\mathrm{d}\vec{\Gamma}}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \vec{r}_i\times m_i\,\vec{v}_i\right)=\sum_i \frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t}\times m_i\,\vec{v}_i+\sum_i \vec{r}_i\times m_i\,\frac{\mathrm{d}\vec{v}_i}{\mathrm{d}t}=$$
$$=\sum_i \vec{v}_i\times m_i\,\vec{v}_i+\sum_i \vec{r}_i\times m_i\,\vec{a}_i=\sum_i \vec{r}_i\times m_i\,\vec{a}_i$$

II. Newtonov zakon za posamezno točkasto maso :

$$m_i\,\vec{a}_i=\sum_j \vec{F}_{ji}+\vec{F}_i$$

$\sum_j \vec{F}_{ji}$ vsota vseh notranjih sil, ki delujejo na maso m_i

$\vec{F}_i$ pa rezultanta vseh zunanjih sil, ki delujejo na maso m_i

$$\frac{\mathrm{d}\vec{\Gamma}}{\mathrm{d}t} = \sum_i \vec{r}_i \times m_i\, \vec{a}_i \qquad\qquad m_i\, \vec{a}_i = \sum_j \vec{F}_{ji} + \vec{F}_i$$

$$\frac{\mathrm{d}\vec{\Gamma}}{\mathrm{d}t} = \sum_i \left(\vec{r}_i \times \sum_j \vec{F}_{ji} \right) + \sum_i \left(\vec{r}_i \times \vec{F}_i \right)$$

KER vsota notranjih navorov $\sum_i \left(\vec{r}_i \times \sum_j \vec{F}_{ji} \right) = 0$

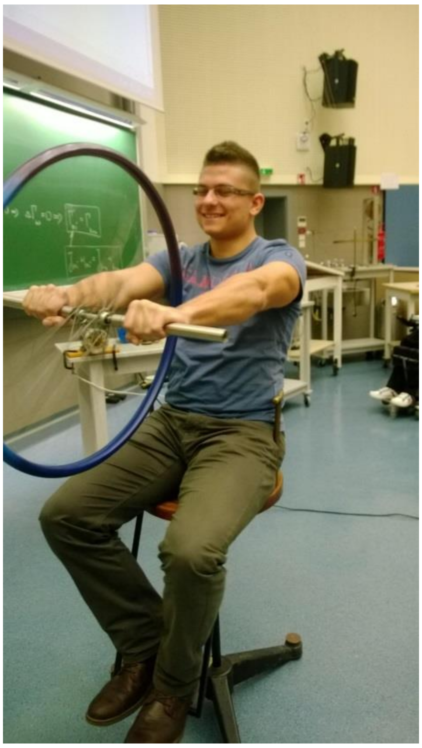
$$\boxed{\frac{\mathrm{d}\vec{\Gamma}}{\mathrm{d}t} = \sum_i \left(\vec{r}_i \times \vec{F}_i \right) = \sum_i \vec{M}_i}$$

izrek o vrtilni količini za sistem točkastih mas

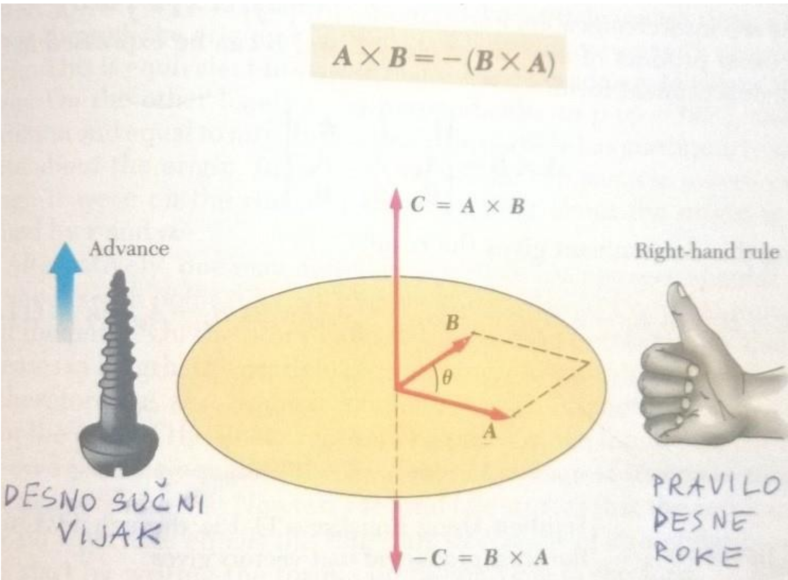
$$\vec{\Gamma} = \sum_i \vec{r}_i \times m_i \vec{v}_i$$

$\sum_i \vec{M}_i$ = vsota vseh zunanjih navorov, ki delujejo na sistem točkastih mas $m_1, m_2, m_3, \ldots$

ZGLED: človek na vrtljivem stolu, z rokami drži kolo od bicikla:

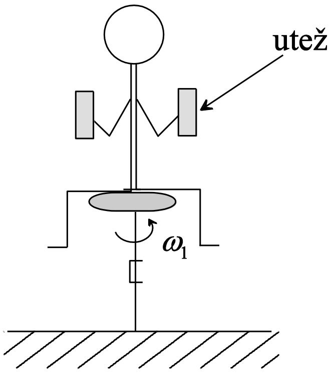
 

$$\vec{\Gamma} = \sum_i \vec{r}_i \times m_i \vec{v}_i \qquad \int \vec{M}\,dt = \Delta\vec{\Gamma} = 0 \qquad \vec{\Gamma} = konst.$$

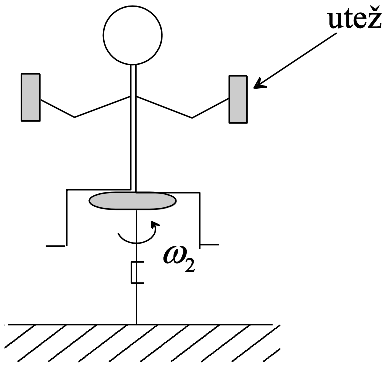


Zgled: človek na vrtljivem stolu z utežmi v obeh rokah (navor notranjih sil = 0)

$$\int_{t_1}^{t_2} \vec{M}\,\mathrm{d}t = 0 \qquad \int_{t_1}^{t_2} \vec{M}\,\mathrm{d}t = J\vec{\omega}_2 - J\vec{\omega}_1 = 0 \qquad J\vec{\omega}_2 = J\vec{\omega}_1$$



$J_1 < J_2$



vztrajnostni moment: J_1
vrtilna količina:
$\Gamma_1 = J_1\,\omega_1$

vztrajnostni moment: J_2
vrtilna količina: $\Gamma_2 = J_2\,\omega_2$

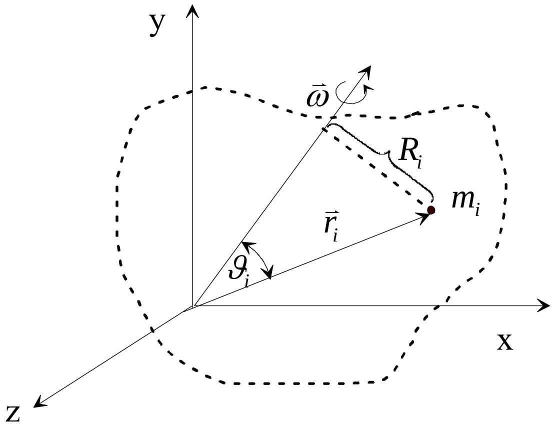
Navor notranjih sil enak nič:

$$J_1\,\omega_1 = J_2\,\omega_2$$

$$\omega_2 = \omega_1 \frac{J_1}{J_2} < \omega_1 \qquad \text{če je : } J_1 < J_2$$

Togo telo

- $\vec{\omega}$ je trenutna kotna hitrost v smeri osi okoli katere se vrtijo vsi deli telesa (vse točkaste mase) z enako kotno hitrostjo: $\vec{\omega}(\vec{r}_i)=\omega=$konst.
- koordinatno izhodišče je na osi vrtenja



$$\vec{\omega}=\left(\omega_x,\omega_y,\omega_z\right)$$

$$\vec{r}_i=\left(x_i,y_i,z_i\right)$$

$$r_i^2=x_i^2+y_i^2+z_i^2$$

$$\vec{v}_i=\vec{\omega}\times\vec{r}_i$$

$$\left|\vec{v}_i\right|=\omega\underbrace{r_i\sin\vartheta_i}_{R_i}=\omega R_i$$

sistem točkastih mas

$$\vec{\Gamma}=\sum_i\vec{r}_i\times m_i\vec{v}_i$$

$$\frac{\mathrm{d}\vec{\Gamma}}{\mathrm{d}t}=\sum_i\vec{M}_i\equiv\vec{M}$$

togo telo: $\sum_i\ \rightarrow\int$, $m_i\rightarrow\mathrm{d}m$, $\vec{r}_i\rightarrow\vec{r}$, $\vec{v}_i\rightarrow\vec{v}$

$$\vec{\Gamma}=\sum_i\vec{r}_i\,x\,m_i\vec{v}_i\ \rightarrow\int\vec{r}\times\vec{v}\,\mathrm{d}m$$

$$\frac{\mathrm{d}\vec{\Gamma}}{\mathrm{d}t}=\vec{M}$$

$$\vec{r}\times\vec{v}=\vec{r}\times\left(\vec{\omega}\times\vec{r}\right)=\vec{\omega}\left(\vec{r}\cdot\vec{r}\right)-\vec{r}\left(\vec{r}\cdot\vec{\omega}\right)$$, od koder sledi:

$$\vec{\Gamma}=\int\vec{r}\times\vec{v}\,\mathrm{d}m=\int\vec{r}\times\left(\vec{\omega}\times\vec{r}\right)\mathrm{d}m=\int\vec{\omega}r^2\mathrm{d}m-\int\vec{r}\,\left(\vec{\omega}\cdot\vec{r}\right)\mathrm{d}m$$

$$\vec{\Gamma}=\int \vec{r}\times\vec{v}\,\mathrm{d}m=\int \vec{r}\times\left(\vec{\omega}\times\vec{r}\right)\mathrm{d}m=\int \vec{\omega}\,r^2\mathrm{d}m-\int \vec{r}\,\left(\vec{\omega}\cdot\vec{r}\right)\mathrm{d}m$$

$$\vec{r}=\left(x,y,z\right) \qquad r^2=\vec{r}\cdot\vec{r}=x^2+y^2+z^2$$

$$\vec{\omega}=\left(\omega_x,\omega_y,\omega_z\right) \qquad \vec{\omega}\cdot\vec{r}=\omega_x x+\omega_y y+\omega_z z$$

zapis po komponentah:

$$\Gamma_x=\omega_x\int r^2\mathrm{d}m-\int x\left(\omega_x x+\omega_y y+\omega_z z\right)\mathrm{d}m$$

$$\Gamma_y=\omega_y\int r^2\mathrm{d}m-\int y\left(\omega_x x+\omega_y y+\omega_z z\right)\mathrm{d}m$$

$$\Gamma_z=\omega_z\int r^2\mathrm{d}m-\int z\left(\omega_x x+\omega_y y+\omega_z z\right)\mathrm{d}m$$

preureditev:

$$\Gamma_x=\omega_x\underbrace{\int\left(y^2+z^2\right)\mathrm{d}m}_{J_{xx}}-\omega_y\underbrace{\int xy\,\mathrm{d}m}_{D_{xy}}-\omega_z\underbrace{\int xz\,\mathrm{d}m}_{D_{xz}}$$

$$\Gamma_y=-\omega_x\underbrace{\int xy\,\mathrm{d}m}_{D_{yx}}+\omega_y\underbrace{\int\left(x^2+z^2\right)\mathrm{d}m}_{J_{yy}}-\omega_z\underbrace{\int zy\,\mathrm{d}m}_{D_{yz}}$$

$$\Gamma_z=-\omega_x\underbrace{\int zx\,\mathrm{d}m}_{D_{zx}}-\omega_y\underbrace{\int zy\,\mathrm{d}m}_{D_{zy}}+\omega_z\underbrace{\int\left(x^2+y^2\right)\mathrm{d}m}_{J_{zz}}$$

vektorski zapis (vektorja $\vec{\Gamma}$ in $\vec{\omega}$ v splošnem **nista** vzporedna):

$$\vec{\Gamma}=\underline{J}\,\vec{\omega}$$

$\underline{J}$ = vztrajnostni tenzor :

$$\underline{J}=\begin{bmatrix} J_{xx} & -D_{xy} & -D_{xz} \\ -D_{yx} & J_{yy} & -D_{yz} \\ -D_{zx} & -D_{zy} & J_{zz} \end{bmatrix}$$

J_{xx}, J_{yy}, J_{zz} so **vztrajnostni momenti** okrog treh pravokotnih osi

$D_{xy}, D_{xz}, D_{yz}, D_{yx}, D_{zx}, D_{zy}$ so **deviacijski momenti**

$$J_{xx}=\int\left(y^2+z^2\right)\mathrm{d}m,$$
$$J_{yy}=\int\left(x^2+z^2\right)\mathrm{d}m,$$
$$J_{zz}=\int\left(x^2+y^2\right)\mathrm{d}m,$$

$$D_{xy}=\int xy\,\mathrm{d}m,\qquad D_{xz}=\int xz\,\mathrm{d}m$$
$$D_{yx}=\int yx\,\mathrm{d}m,\qquad D_{yz}=\int yz\,\mathrm{d}m$$
$$D_{zx}=\int zx\,\mathrm{d}m,\qquad D_{zy}=\int yz\,\mathrm{d}m$$

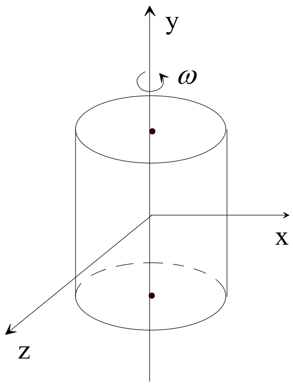
$$D_{xy}=D_{yx}$$
$$D_{xz}=D_{zx}$$
$$D_{yz}=D_{zy}$$

Diagonalizacija tenzorja $\underline{J}$: $D_{ij}=0$, koordinate osi so usmerjene vzdolž glavnih osi:

$$\underline{J}=\begin{vmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{vmatrix},\quad J_x\equiv J_{xx},\ J_y\equiv J_{yy},\ J_z\equiv J_{zz}$$

Poseben primer: $\vec{\omega}=(0,\omega,0)$ v sistemu glavnih osi ($\underline{J}$ je diagonaliziran)

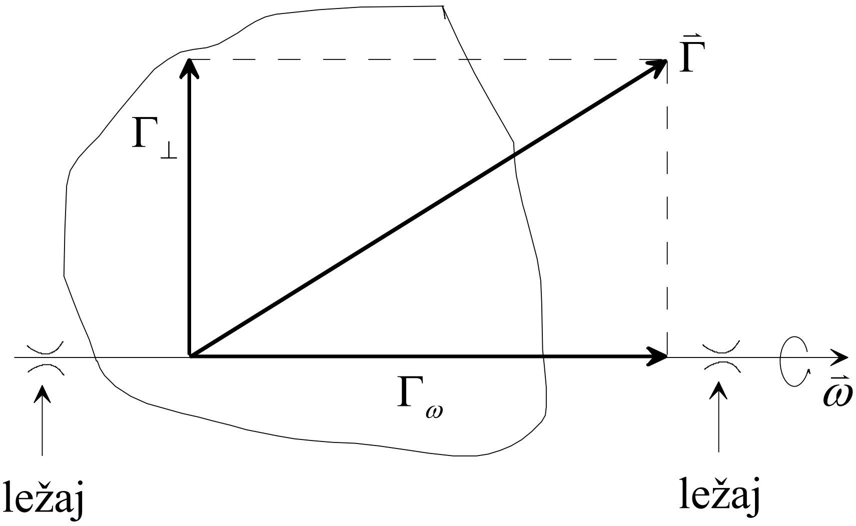
$\vec{\Gamma}$ in $\vec{\omega}$ sta vzporedna

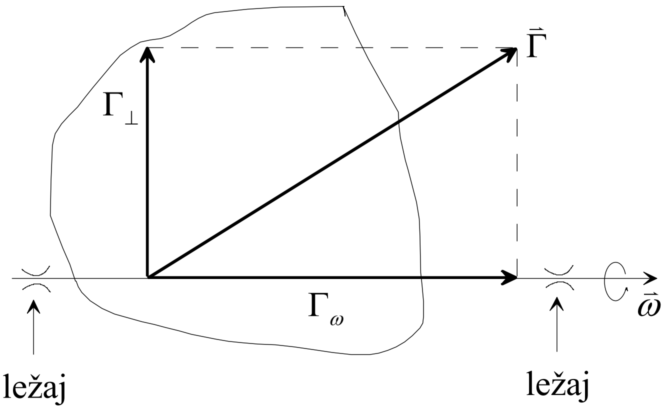


$$\Gamma_y=J_y\,\omega,$$

$$\frac{\mathrm{d}\Gamma_y}{\mathrm{d}t}=M_y \qquad\qquad J_y=\int\left(x^2+z^2\right)\mathrm{d}m$$

Poseben primer: **os vrtenja togega telesa je vpeta v ležaje**





Komponenta vektorja $\vec{\Gamma}$ v smeri kotne hitrosti $\vec{\omega}$

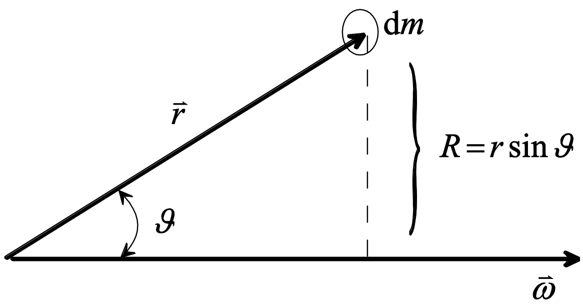
$$\Gamma_\omega = \vec{\Gamma} \cdot \frac{\vec{\omega}}{\omega} \qquad \left(\left| \frac{\vec{\omega}}{\omega} \right| = 1 \right)$$

$M_\omega =$ komponento navora $\vec{M}$ v smeri vektorja kotne $\vec{\omega}$:

$$\Gamma_\omega = \frac{\vec{\omega}}{\omega} \cdot \int \vec{r} \times \vec{v}\, \mathrm{d}m = \frac{\vec{\omega}}{\omega} \cdot \int \vec{r} \times (\vec{\omega} \times \vec{r})\, \mathrm{d}m = \frac{\vec{\omega}}{\omega} \cdot \left[\int r^2 \vec{\omega}\, \mathrm{d}m - \int \vec{r}\, (\vec{r} \cdot \vec{\omega})\, \mathrm{d}m \right] =$$

$$= \int \left(r^2 \omega - \frac{(\vec{r} \cdot \vec{\omega})^2}{\omega} \right) \mathrm{d}m = \int r^2 \omega \left(1 - \cos^2 \vartheta \right) \mathrm{d}m = \int r^2 \omega \sin^2 \vartheta = \omega \int R^2\, \mathrm{d}m = \omega J_\omega$$

$$\vec{r} \cdot \vec{\omega} = r\, \omega \cos \vartheta \qquad R = r \sin \vartheta \qquad J_\omega = \int R^2\, \mathrm{d}m$$



$$\Gamma_\omega = J_\omega\, \omega \qquad M_\omega = \frac{\mathrm{d}\Gamma_\omega}{\mathrm{d}t} = J_\omega \frac{\mathrm{d}\omega}{\mathrm{d}t} = J_\omega\, \alpha$$

$$M_\omega = J_\omega \alpha \qquad \alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \text{ kotni pospešek}$$

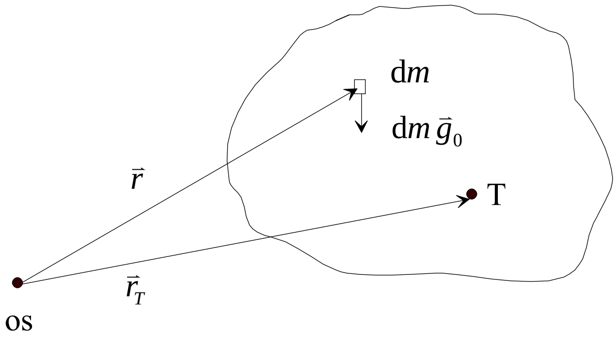
$$M_\omega = \frac{\mathrm{d}\Gamma_\omega}{\mathrm{d}t} \qquad \int M_\omega \,\mathrm{d}t = \int_{\Gamma_1}^{\Gamma_2} d\,\Gamma_\omega = \Gamma_2 - \Gamma_1 = \Delta\Gamma$$

$$\int M_\omega \,\mathrm{d}t = \Delta\Gamma = 0 \qquad \Gamma_2 = \Gamma_1$$

Poskus: tresenje v ležajih

Zgled: vreteno in utež

Navor sile teže telesa



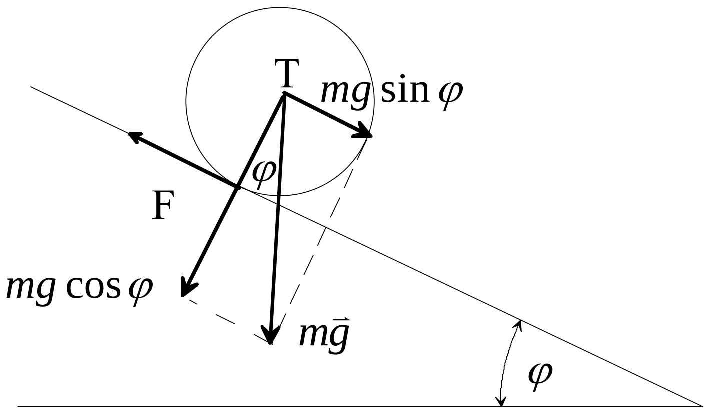
T ≡ težišče telesa
dm ≡ masa majhnega koščka togega telesa

$$\vec{M}_g = \int \vec{r} \times \mathrm{d}\vec{F}_g = \int \vec{r} \times \mathrm{d}m\,\vec{g}_0 = \int \vec{r}\,\mathrm{d}m \times \vec{g}_0 = \left(\int \vec{r}\,\mathrm{d}m\right) \times \vec{g}_0 =$$

$$= m\frac{\int \vec{r}\,\mathrm{d}m}{m} \times \vec{g}_0 = m\,\vec{r}_T \times \vec{g}_0 = \vec{r}_T \times m\vec{g}_0 = \vec{r}_T \times \vec{F}_g$$

$$\vec{r}_T = \frac{1}{m}\int \vec{r}\,\mathrm{d}m \qquad\qquad \vec{F}_g = \int \mathrm{d}m\,\vec{g}_0 = m\,\vec{g}_0$$

Zgled: kotaljenje **homogenega valja** po klancu navzdol **brez podrsavanja**



$$mg\sin\varphi - F = m\,a_T$$

$$FR = J\,\alpha = \frac{1}{2}m\,R^2\,\frac{a_T}{R} = \frac{1}{2}R\,m\,a_T$$

kotaljenje brez podrsavanja je hitrost težišča valja

$$J = \frac{1}{2}m\,R^2 \qquad v_T = R\,\omega \qquad a_T = R\,\alpha$$

$$mg\sin\varphi - \frac{1}{2}m\,a_T = m\,a_T$$

$$a_T = \frac{2}{3}g\sin\varphi$$

$$F = \frac{1}{2}m\,a_T = \frac{1}{3}m\,g\sin\varphi$$

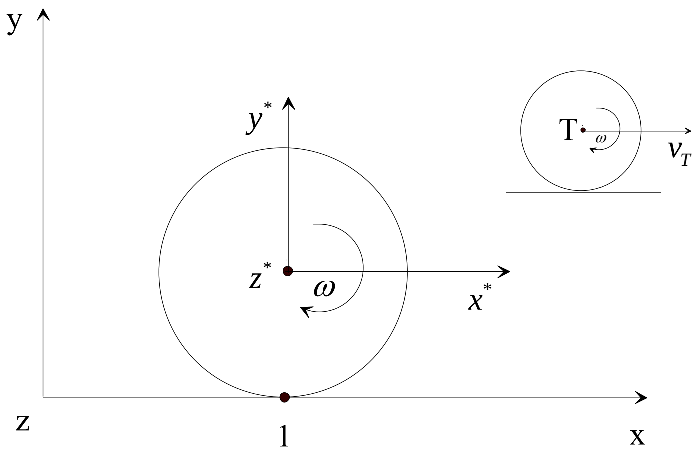
pogoj za »**nepodrsavanje**« : $F = \frac{1}{3}m\,g\sin\varphi < F_\ell = m\,g\cos\varphi\,k_\ell$

$$tg\,\varphi < 3k_\ell$$

Poskus: kotaljenje po klancu: tekma valjev (poln in votel valj na strmini)

$$a_T = \frac{2}{3} g \sin\varphi \qquad\qquad a_T = \frac{1}{2} g \sin\varphi$$

Dokaz veljavnosti enačbe: $v_T = R\omega$ (**brez podrsavanja**)



hitrost točke 1 v laboratorijskem sistemu: $v_1 = 0$

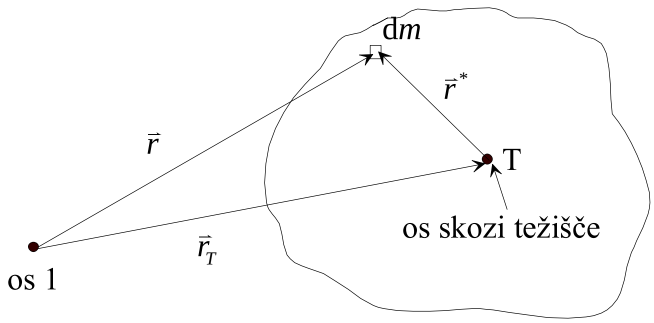
hitrost točke 1 merjena v težiščnem sistemu: $v_1^* = -R\omega$

R = polmer preseka valja

Galilejeva transformacija za hitrosti: $v_1 = v_1^* + v_T$

$$v_T = -v_1^* = R\omega$$

Steiner-jev izrek (povezuje vztrajnostni moment 2 togih teles okoli dveh paralelnih osi, od katerih poteka ena skozi težišče telesa)



T ≡ težišče telesa

dm ≡ masa majhnega dela togega telesa

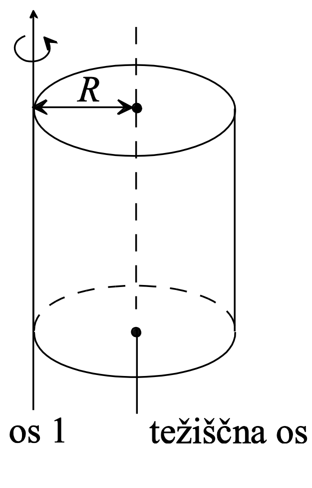
$\vec{r} = \vec{r}_T + \vec{r}^*$

$$J_1 = \int r^2 \mathrm{d}m = \int \vec{r} \cdot \vec{r}\, \mathrm{d}m = \int \left(\vec{r}_T + \vec{r}^*\right) \cdot \left(\vec{r}_T + \vec{r}^*\right) \mathrm{d}m =$$

$$= \int r_T^2 \mathrm{d}m + 2\vec{r}_T \cdot \int \vec{r}^* \mathrm{d}m + \int r^{*2} \mathrm{d}m = m r_T^2 + J^*$$

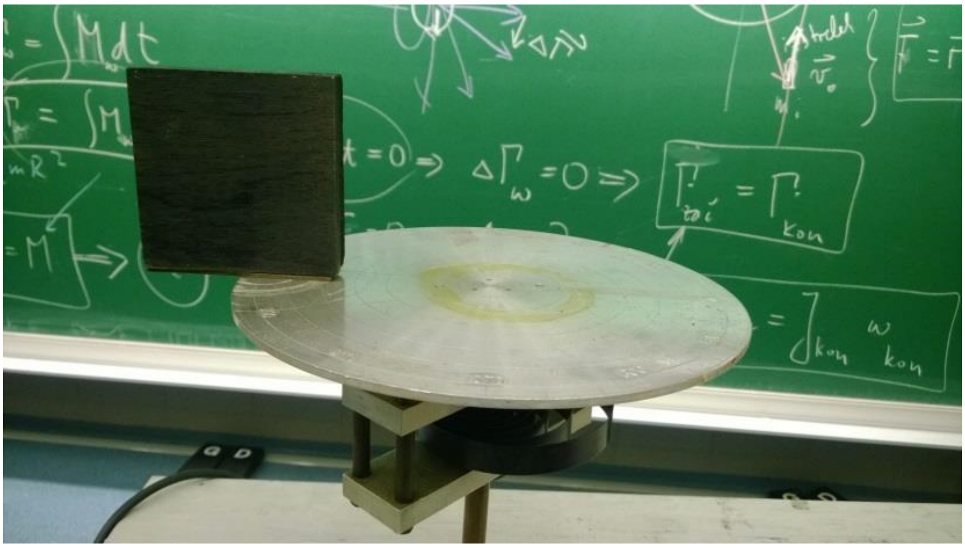
$$\vec{r}_T^* = \frac{1}{m} \int \vec{r}^* \mathrm{d}m = 0$$

Primer:



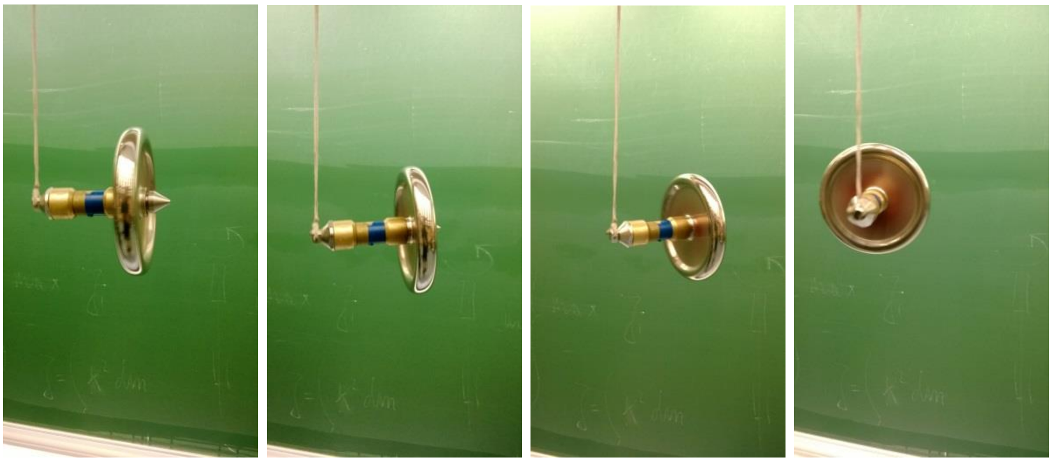
$$J_1 = J^* + m r_T^2 = \frac{1}{2} m R^2 + m R^2 = \frac{3}{2} m R^2 .$$

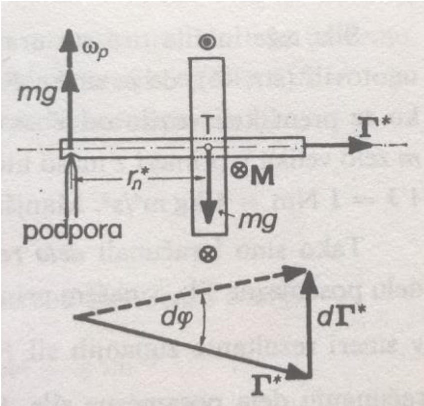
Poskus: streljanje v tarčo na mizici sučnega balističnega nihala



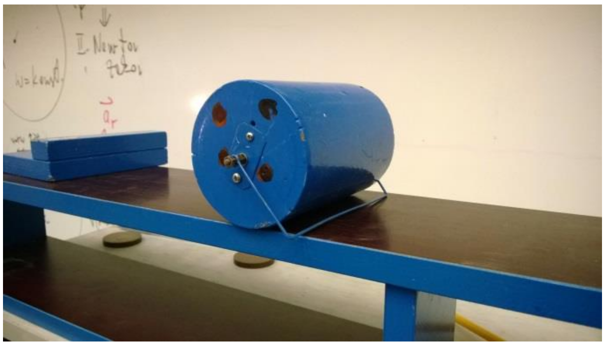
Poskus: mali bicikel

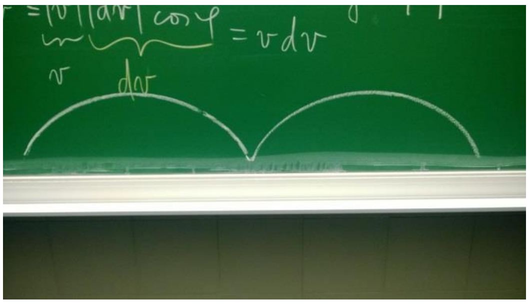
Poskus: precesija (vrtavka na vrvici)





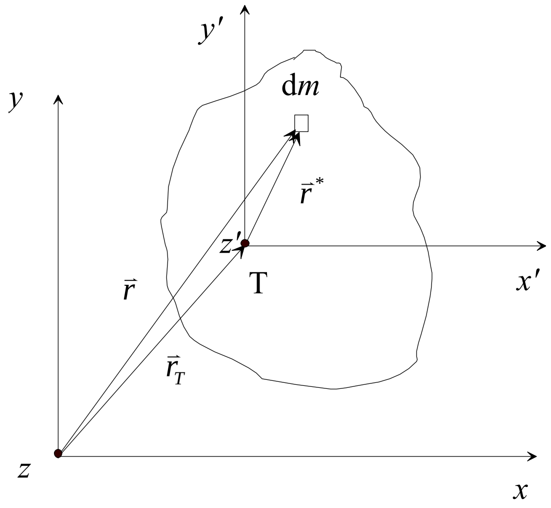
Kinetična energija togega telesa v splošnem





- izraz $\frac{1}{2}mv^2$ le, če se togo telo giblje po premici

predpostavimo, da so koordinatne osi laboratorijskega inercialnega koordinatnega sistema (x, y, z) paroma vzporedne z osmi lokalnega težiščnega sistema (x', y', z') :



T ≡ težišče togega telesa

$$\vec{r} = \vec{r}_T + \vec{r}^{\,*} \qquad \vec{v} = \vec{v}_T + \vec{v}^{\,*}$$

$$W_k = \frac{1}{2}\int v^2\,\mathrm{d}m = \frac{1}{2}\int \vec{v}\cdot\vec{v}\,\mathrm{d}m$$

$$W_k = \frac{1}{2}\int \left(\vec{v}_T + \vec{v}^{\,*}\right)\cdot\left(\vec{v}_T + \vec{v}^{\,*}\right)\mathrm{d}m = \frac{1}{2}\int v_T^2\mathrm{d}m + v_T\cdot\int \vec{v}^{\,*}\,\mathrm{d}m + \frac{1}{2}\int \vec{v}^{\,*2}\,\mathrm{d}m$$

$$\vec{r}_T^{\,*} = \frac{1}{m}\int \vec{r}^{\,*}\,\mathrm{d}m = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{m}\int \vec{r}^{\,*}\,\mathrm{d}m\right) = \frac{1}{m}\int \vec{v}^{\,*}\mathrm{d}m = 0$$

vrtenje okoli trenutne osi s kotno hitrostjo ω, ki je v izbranem trenutku enaka za vse točkaste mase dm : $v^* = R^*\omega$

$$W_k = \frac{1}{2}mv_T^2 + \frac{1}{2}J^*\,\omega^2$$

$$\frac{1}{2}\int v^{*2}\,\mathrm{d}m = \frac{1}{2}\omega^2\int R^{*2}\mathrm{d}m = \frac{1}{2}J^*\omega^2 \qquad J^* = \int R^{*2}\,\mathrm{d}m$$

za izbrani težiščni sistem lahko v splošnem zapišemo rotacijsko kinetično energijo kot:

$$W_r = \frac{1}{2} \sum J_{\alpha\beta}\, \omega_\alpha\, \omega_\beta$$

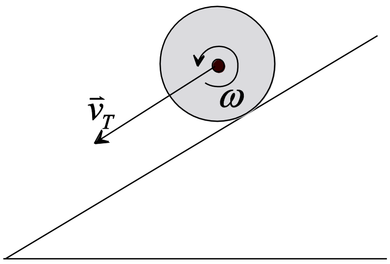
$$\alpha = x^*, y^*, z^*, \quad \beta = x^*, y^*, z^*$$

kjer so $J_{x^*x^*}, J_{y^*y^*}$ in $J_{z^*z^*}$ vztrajnostni momenti okrog treh pravokotnih osi x^*, y^* in z^*, $J_{\alpha\beta}(\alpha \neq \beta)$ pa so deviacijski momenti. Če so osi težiščnega sistema hkrati tudi glavne osi dobi izraz (29) obliko:

$$W_r = \frac{1}{2} J_{x^*}\, \omega_1^2 + \frac{1}{2} J_{y^*}\, \omega_2^2 + \frac{1}{2} J_{z^*}\, \omega_3^2$$

Primer : če se trenutna os vrtenja vedno ujema z eno izmed glavnih osi, n.pr. z J_{x^*}

$$W_r = \frac{1}{2} J_{x^*} \omega_1^2$$



$$v_T = R\,\omega \qquad\qquad J = \frac{1}{2} m R^2$$

$$W_k = \frac{1}{2} m v_T^2 + \frac{1}{2} J\, \omega^2 = \frac{1}{2} m v_T^2 + \frac{1}{2}\left(\frac{1}{2} m R^2\right)\frac{v_T^2}{R^2} = \frac{3}{4} m v_T^2$$

Zgled: vrtenje okoli ene izmed glavnih osi

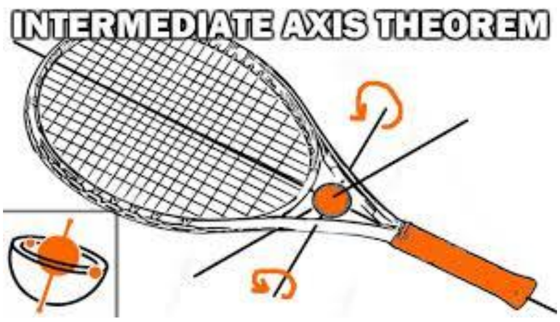
če se trenutna os vrtenja vedno ujema z eno izmed glavnih osi :

okoli x-osi: $$W_r = \frac{1}{2} J_x \, \omega_1^2 = \frac{\Gamma^2}{2J_x}$$

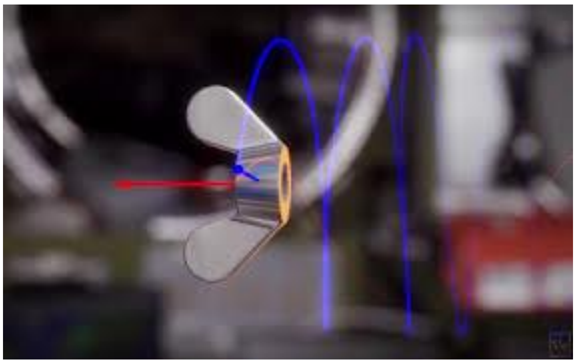
okoli y-osi: $$W_r = \frac{1}{2} J_y \, \omega_2^2 = \frac{\Gamma^2}{2J_y}$$

okoli z-osi: $$W_r = \frac{1}{2} J_z \, \omega_3^2 = \frac{\Gamma^2}{2J_z}$$

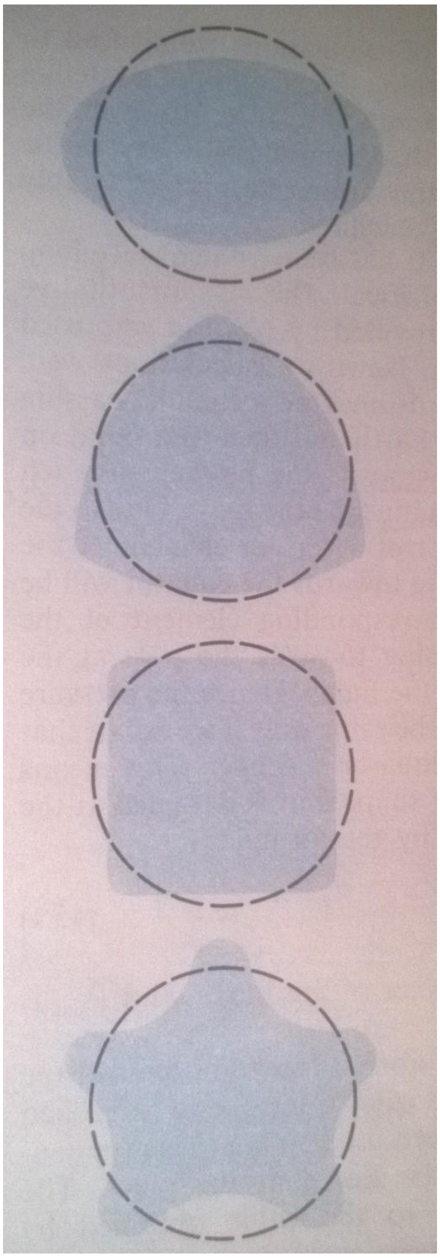
Dzhanibekov efekt



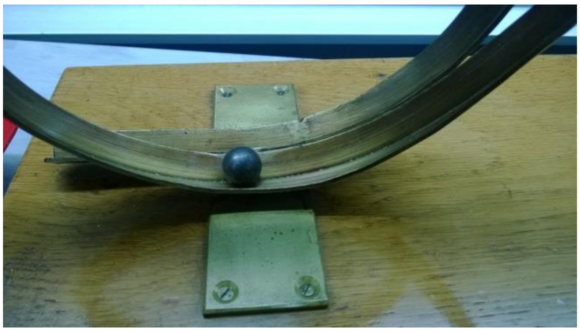
https://www.youtube.com/watch?v=1VPfZ_XzisU

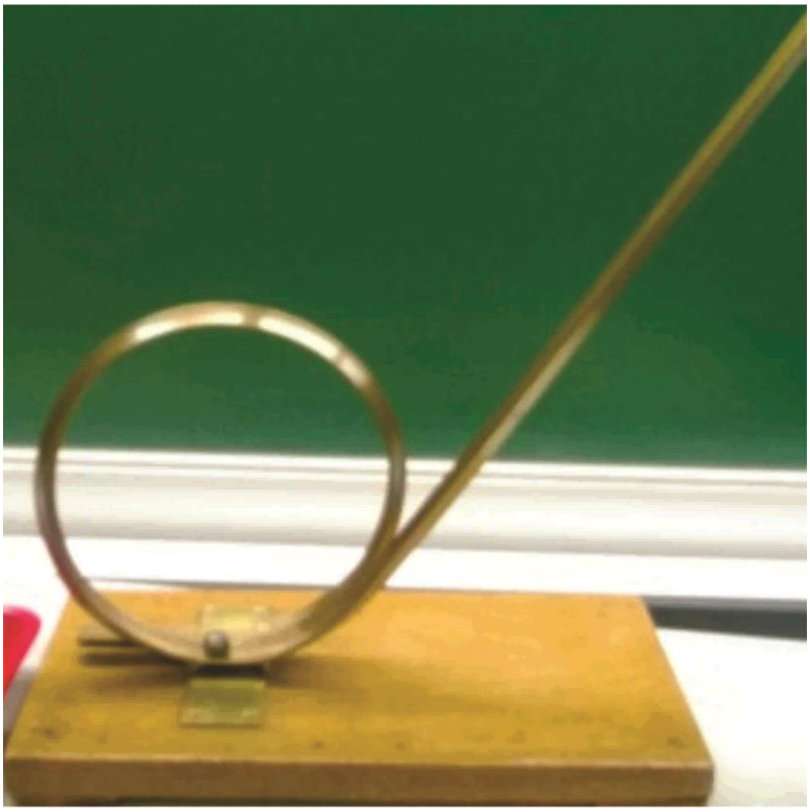


oblika Zemlje :



ZGLED: »LOOPING« - zaviti žleb in kovinska kroglica, ohranitev vsote kinetične in gravitacijske potencialne energije





MEHANSKO RAVNOVESJE

$$\sum_i \vec{F}_i = 0 \qquad \sum_i \vec{M}_i = 0$$

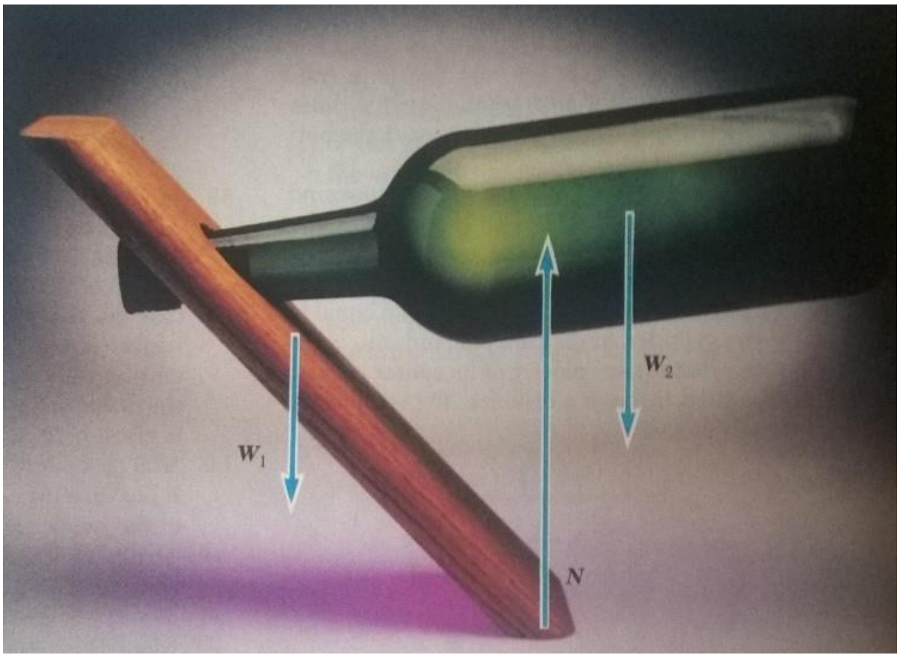


Foto: Serway