STRENGTH OF MATERIALS II

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Preface

This textbook aims to help students taking courses taught in English at CTU, Faculty of Mechanical Engineering, in their studies of one of the most important, and, at the same time, most difficult engineering subjects. This course will be taken not only by foreign students (speaking good English and knowing English technical and mathematical terms), but also by Czech students intending to improve their English while studying a professional subject (perhaps with a view to continuing their studies abroad).

This textbook, entitled "Strength of Materials II", links up with "Strength of Materials I" (published 1999), and is intended as support material for the 2nd course on "Strength of Materials". It was not easy to prepare these study materials, due to extraordinary limitation of space, and to avoid simply presenting formulas together with brief comments. An attempt was made to preserve both pedagogical values (professional and linguistic) and comprehensive coverage of the prescribed study material. Much of the textbook is devoted to a description or characterization of problems, especially in the first part of each chapter. For reasons explained above, it has not been possible to discuss all suitable methods, and only a relatively small number of examples are presented.

The author is indebted to Mr. Robin Healey, who promptly and willingly reviewed the text, and to Ing. Pavel Šídlo and Mr. Petr Tichý, who made most of the figures and prepared the text for publication. The author is repeatedly indebted to his wife, Dr. Ludmila Sochorová, for her patience and understanding during the preparation of the manuscript.

1. Curved and cranked rods and frames

Applying the *method of sections* on a curved rod (CR) we observe 3 internal force actions: *normal force* N_A, *shearing force* V_A, and *bending moment* M_A, Fig. 1.1. The analysis of curved rods depends on the ratio of their radius of curvature r to its depth d:

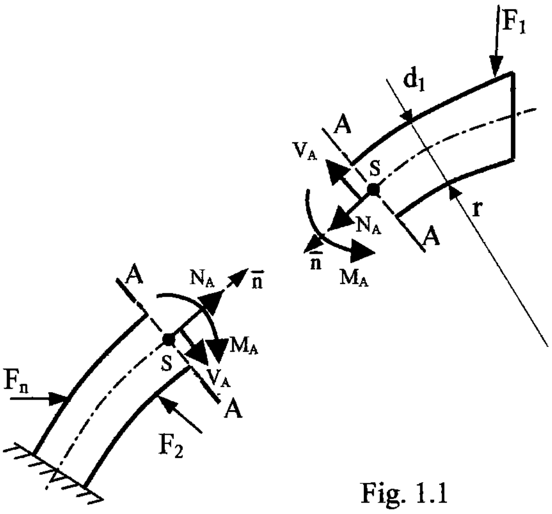


Fig. 1.1

1) *Thick CR*: $\frac{r}{d} \leq 5$; 2) *Thin CR*: $\frac{r}{d} > 5$

The analysis of a *thick CR* will not be discussed in this textbook.

When turning our attention to the problems of a *thin CR (TnCR)* we can treat it as a *beam*, i.e., only a strength criterion for *bending* will be applied on *TnCR* (the influence of N_A and V_A will be neglected).

Note: The neutral axis of a cross-section will coincide with its respective central principal axis.

1.1 Statically determinate *thin curved rods* and *cranked* rods

1.1.1 Displacements of *TnCR*

In case of TnCR, an application of *Mohr's integral* (cf. [1], Chap.11, Eq.11.3.1.1*b*), based on *Castigliano's theorem*, is suitable, e.g., to assess displacements in *TnCR* at point *B* (Fig.1.1.1.1): ($M(\xi) = F.\ r.\ \sin\varphi$... bending moment caused by given load F)

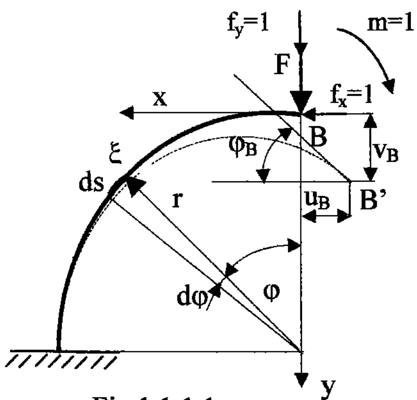


Fig.1.1.1.1

$u_B = \frac{1}{EI}\int_L M(\xi)\cdot m_x(\xi)\cdot ds$; $m_x(\xi) = -r\cdot(1-\cos\varphi)$ ... *b.m. caused by dummy load* $\mathbf{f_x}$

$v_B = \frac{1}{EI}\int_L M(\xi)\cdot m_y(\xi)\cdot ds$; $m_y(\xi) = r\cdot\sin\varphi$ ... *b.m. caused by dummy load* $\mathbf{f_y}$

$\Delta\varphi_B = \frac{1}{EI}\int_L M(\xi)\cdot m(\xi)\cdot ds$; $m(\xi) = 1$ ... *b.m. caused by dummy couple* **m** (1.1.1.1*a,b,c*)

Results: $u_B = -\frac{Fr^3}{2EI}$; $v_B = \frac{\pi Fr^3}{4EI}$; $\Delta\varphi_B = \frac{Fr^2}{EI}$

1.1.2 Displacements of a cranked rod

In the case of *cranked rods*, an application of *Vereščagin's rule* (cf. [1], Chap.11, Eq.11.3.1.1*b*), a modification of *Mohr's integral,* is suitable, e.g., to assess displacements in *a cranked rod* at point *A* (Fig.1.1.2.1): $u_A = \frac{1}{EI}\sum_{i=n} A_M \cdot m_{xC}$; $v_A = \frac{1}{EI}\sum_{i=n} A_M \cdot m_{yC}$; $\Delta\varphi_A = \frac{1}{EI}\sum_{i=n} A_M \cdot m_C$ (1.1.2.1*a,b,c*)

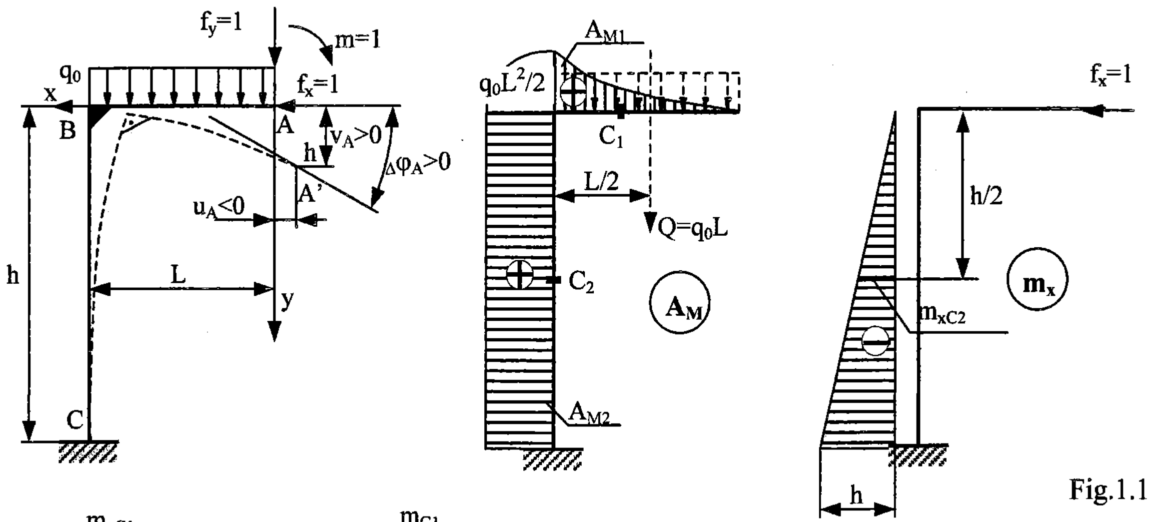
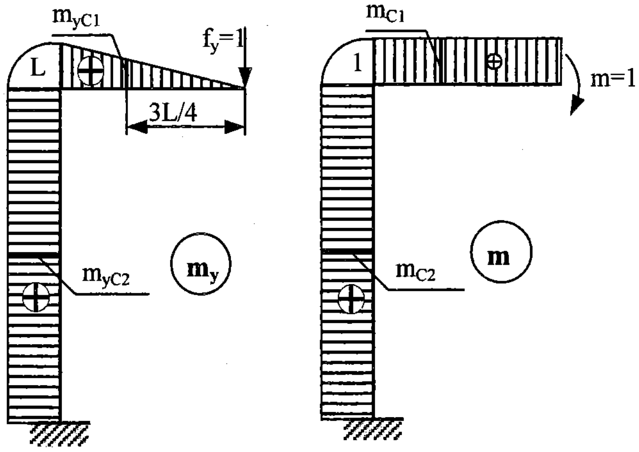


Fig.1.1.2.1



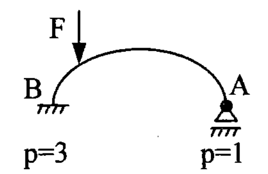
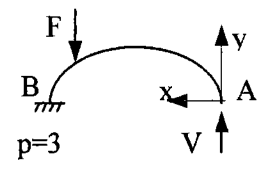
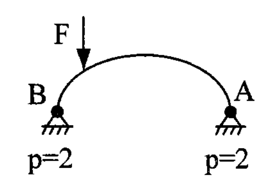
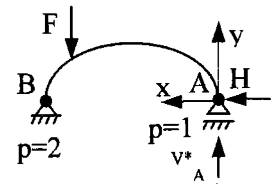
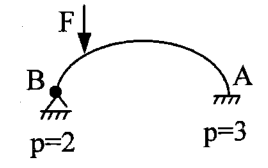
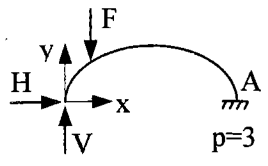
$$u_A = \frac{1}{EJ}\left[\frac{q_0L^2}{2}L\frac{1}{3}0 + \frac{q_0L^2}{2}h\left(-\frac{h}{2}\right)\right] = -\frac{q_0L^2h^2}{4EJ}$$

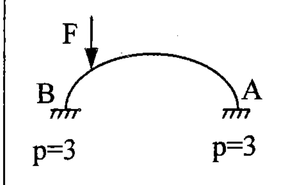
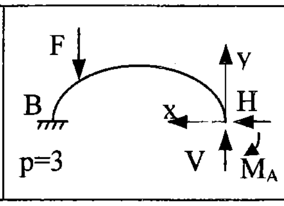
$$v_A = \frac{1}{EJ}\left[\frac{q_0L^2}{2}L\frac{1}{3}\frac{3}{4}L + \frac{q_0L^2}{2}hL\right] = \frac{q_0L^3}{4EJ}\left(\frac{L}{2} + 2h\right)$$

$$_{\Delta}\varphi_A = \frac{1}{EJ}\left[\frac{q_0L^2}{2}L\frac{1}{3}1 + \frac{q_0L^2}{2}h1\right] = \frac{q_0L^2}{2EJ}\left(\frac{L}{3} + h\right)$$

1.2 Statically indeterminate *thin curved rods* and *cranked* rods

Note: SI curved and cranked rods can be solved by the procedure shown in the following table: 1) °SI determination; 2) Releasing to SD basic system; 3) Application of the respective number of compatibility eqs. expressed either by Mohr's integral or by Vereščagin's rule.

°SI	Σp	Rod supports	Releasing to SD basic system	Compatibility equations
1	4			$v_A=0$
1	4			$u_A=0$ **Note**: Here it is necessary to express the moment eq. about pin *B*: $M_B = 0$
2	5			$u_B=0$ $v_B=0$

3	6			$u_A=0$ $v_A=0$ $\Delta\varphi_A=0$

1.3 Statically indeterminate frames

Closed frames are statically indeterminate to the third degree (3°SI), Fig.1.3.1, which can be assessed by applying the *method of sections.* The problem can be restated as the solution of a statically indeterminate (*3°SI*) *thin curved* (or *cranked*) *rod,* Fig.1.3.2., one end of which being as if fixed (left) and the second as if free.

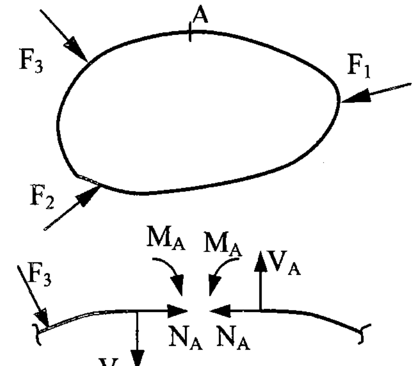


Fig.1.3.1.

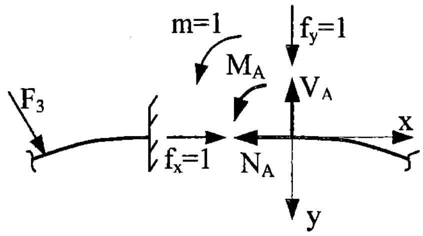


Fig.1.3.2

If a frame is symmetrical <u>both *geometrically* and with respect to its *load*</u>, this symmetry can be utilized, and the degree of static indeterminateness thus decreases:

***One** axis of symmetry*: <u>Shearing force V_A vanishes</u> ⇒ the frame is *2° SI* (M_A and N_A remain)

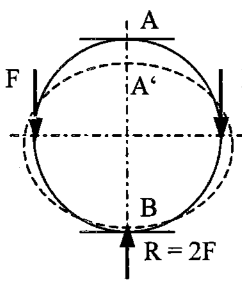
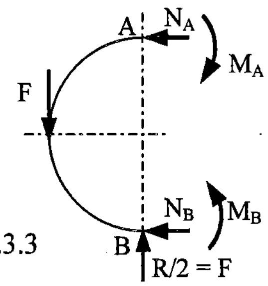
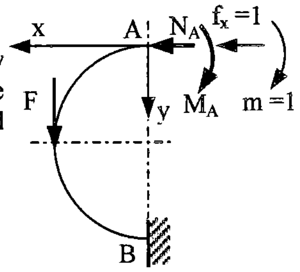


Fig. 1.3.3



Due to the symmetry only one half of the frame has to be solved

Comp. eqs.:
$\Delta\varphi_A = 0$
$u_A = 0$



***Two** axes of symmetry*: <u>Shearing force V_A vanishes and normal force N_A is obtained from a force equilibrium equation</u> ⇒ the frame is *1° SI* (M_A remains):

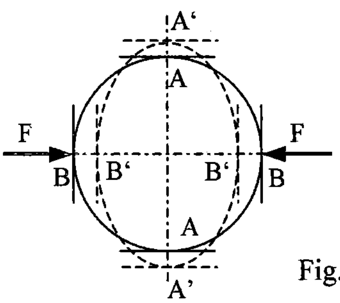
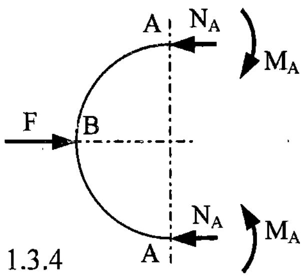
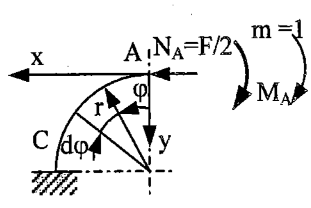


Fig. 1.3.4



Due to the double symmetry only one quarter of the frame has to be solved

Comp. eq.:
$\Delta\varphi_A = 0$



2. Axially (rotationally) symmetric problems in the theory of elasticity

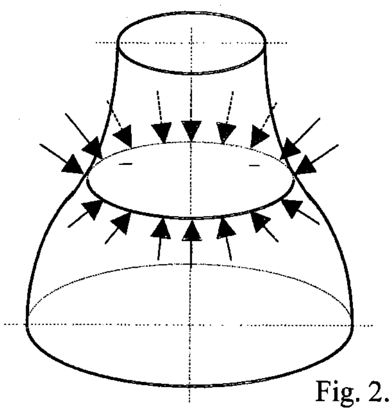


Fig. 2.1

Rotationally symmetric members can be shown to be structures which are rotationally symmetric not only with respect to their shapes but also with respect to their loadings and the resulting displacements and stresses.

Note: Rotationally symmetric tasks that we will deal with involve *thick cylinders (TC)*, and *circular plates (CP)*. Necessary physical quantities and their relations can be derived (generally for all rotationally symmetric tasks, i.e., *thick cylinders*, *circular plates* and also *rotating disks*) when following a procedure (shown below) divided into 9 steps. In Sec. 2.1, we will apply them to *thick cylinders*, while in Sec. 2.2, we will apply similar steps to *circular plates*.

2.1 Thick cylinders (vessels)

Thick cylinders can be divided with respect to their type of stress state (cf. Sec. 2.1.1):

a) **Closed vessels** (e.g., pressure vessels) - main feature: ***axial stress* arises**, which results in 3D-stress state

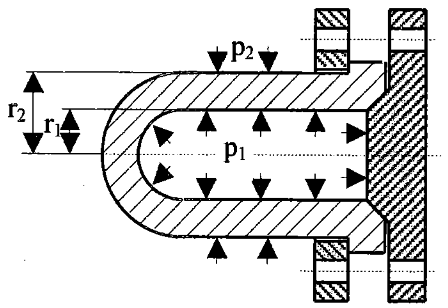


Fig. 2.1.1

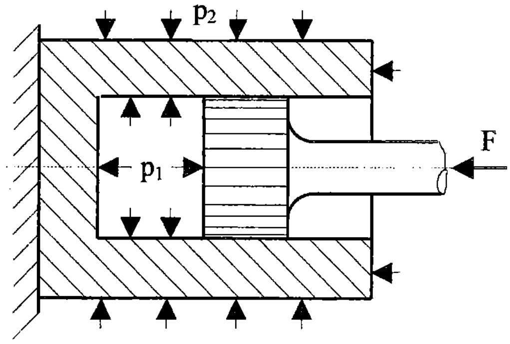


Fig.2.1.2

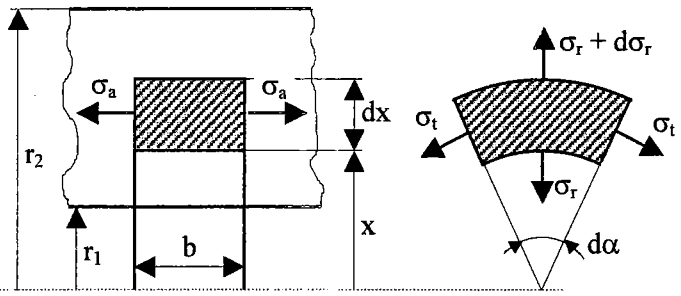
b) **Open vessels** (e.g., engine cylinder with piston, hydraulic cylinder, gun barrel) - main feature: **without *axial stress***, which results in a 2D-stress state

2.1.1 Elastic stress formulas

Elastic stress formulas for thick cylinders under uniform pressure applied on both the inner and outer cylinder surfaces will be derived. The following formulas are applicable for sections some distance from the ends of the cylinder, where the effects of end constraints are negligible, according to Saint-Venant's Principle (see Chap. 1 in [1]). The applied loadings (pressures), as well as the resulting displacements and stresses are axially symmetric. The solution is based on a generalized plane strain model for which both the stress (if any) and the strain, arising in the cylinder axial direction, are constant.

1) **Equilibrium equation**

This can be obtained by using a segmental element. In Fig. **1**/2.1.1*a*, both positive *radial* stresses σ_r and *tangent* (*circumferential, hoop*) stresses σ_t are plotted.



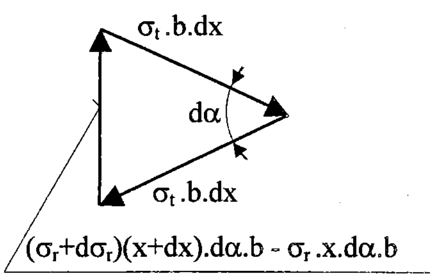


Fig. 1/2.1.1*a* Fig. **1**/2.1.1*b*

Expressing the internal forces produced by the stresses exerting on the respective element surfaces, we can draw their equilibrium (balance) triangle (Fig. 1/2.1.1b) from which we can write the following equilibrium equation

$$d(\sigma_r \cdot x) - \sigma_t \cdot dx = 0, \quad \text{or} \quad \frac{d\sigma_r}{dx} \cdot x + \sigma_r - \sigma_t = 0 \qquad \textbf{(1/2.1.1}\textit{a,b}\textbf{)}$$

We can see that there are two unknown stresses and only one equilibrium equation, from which it follows that the problem is internally statically indeterminate (to the 1° degree). This means that we must add (define) one *compatibility equation* (*deformation condition*) which can be obtained by defining the relation between *displacements* and *strains* of the cylinder segment.

2) **Relation between *displacements* and *strains***

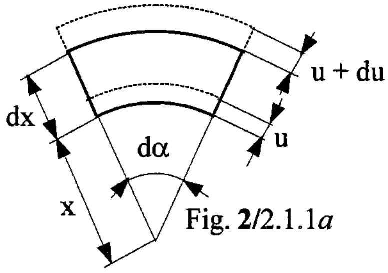


Fig. **2**/2.1.1*a*

We shall start from a possible element deformation given by an increment (displacement) $\boldsymbol{u}$ of an arbitrary radius $\boldsymbol{x}$ of the cylinder and an increment $\boldsymbol{u + du}$ of the infinitesimally increased radius $\boldsymbol{x + dx}$ (Fig. **2**/2.1.1*a*), taking into account that, due to the rotational symmetry of the problem, the element circumferential coordinate **dα** cannot change. Then, with respect to the fact that the *strain* is defined as the ***ratio***

$$\frac{\text{new element length - original element length}}{\text{original element length}},$$ we shall compute both the *tangent* (circumferential, hoop) and *radial* strains and obtain successively:

$$\varepsilon_t = \frac{(x+u)\cdot d\alpha - x \cdot d\alpha}{x \cdot d\alpha} = \frac{u}{x} \qquad (2/2.1.1a)$$

$$\varepsilon_r = \frac{[dx + (u+du) - u] - dx}{dx} = \frac{du}{dx} = u' \qquad (2/2.1.1b)$$

Comparing the resulting expressions from the first two steps we can see that they are not, evidently, compatible. This means that *constitutive relations* are necessary. For that reason we use the *generalized Hooke's law.*

3) **Generalized Hooke's Law**

When solving a **closed vessel** problem, we need to deal with a **3D-stress** problem (i.e., σ_r - *radial* stress, σ_t - *tangent* stress, and σ_a - *axial* stress). On the other hand, when solving the **open vessel** problem, we deal only with a **2D-stress** problem (σ_r - *radial* stress and σ_t - *tangent* stress). The 3D-stress problem is substantially more difficult to solve than the 2D-stress problem, and, surprisingly, the final result, concerning the basic thick cylinder differential equations for the two cylinder types are the same, as can be proved. In our course we shall apply the simpler 2D-approach.

We express the generalized Hooke's law in three modes:

a) Referring to Sec.5.8 in [1], we express the corresponding strains:

The *tangent* strain $$\varepsilon_t = \frac{u}{x} = \frac{1}{E} \cdot [\sigma_t - \mu \cdot \sigma_r] \qquad (3/2.1.1a)$$

The *radial* strain $$\varepsilon_r = u' = \frac{1}{E} \cdot [\sigma_r - \mu \cdot \sigma_t] \qquad (3/2.1.1b)$$

b) Then we rearrange them into a form needed for substitution into <u>the first step - *the differential equilibrium equation*</u>, i.e., expressing the corresponding stresses (when applying Eqs. **2**/2.1.1*a,b,* we express the stresses as functions of displacement u):

The *tangent* stress $$\sigma_t = \frac{E}{1-\mu^2} \cdot [\varepsilon_t + \mu \cdot \varepsilon_r] = E^x \cdot \left[\frac{u}{x} + \mu \cdot u'\right] \qquad (3/2.1.1c)$$

The *radial* stress $$\sigma_r = \frac{E}{1-\mu^2} \cdot [\varepsilon_r + \mu \cdot \varepsilon_t] = E^x \cdot \left[u' + \mu \cdot \frac{u}{x}\right] \qquad (3/2.1.1d)$$

c) Finally we differentiate the relation for the radial stress having

$$\frac{d\sigma_r}{dx} = E^x \cdot \left[u'' + \mu \cdot \frac{u' \cdot x - u}{x^2}\right] \qquad (3/2.1.1e)$$

which we also need for substitution into *the first step.*

4) **The basic thick cylinder differential equation** (BTCDE) (which we obtain after substituting Eqs.3/2.1.1*c,d,e* into Eq.1/2.1.1*b*):

$$x \cdot u'' + u' - \frac{u}{x} = 0 \qquad (4/2.1.1a)$$

This step completes the principal (invariant) part of the solution. A similar approach and procedure can be applied to rotational disks (we are not going to study them in our course) and circular (Kirchhoff's) plates (see Sec. 2.2).

5) **Solution of the BTCDE**

BTCDE, a differential equation of the Euler type, has a particular solution in the form

$$u = x^n \qquad (5/2.1.1.a)$$

which, after substituting all its required derivatives ($u' = n.x^{n-1}$; $u'' = n.(n-1).x^{n-2}$) into Eq.4/2.1.1*a*, will result in (while $x \neq 0$):

$$x^{n-1} \cdot [n \cdot (n-1) + n - 1] = 0 \quad \Rightarrow \quad n = \pm 1 \qquad (5/2.1.1b)$$

A general solution of BTCDE is

$$u = C_1 \cdot x + \frac{C_2}{x}; \; \left(u' = C_1 - \frac{C_2}{x^2} \ldots \text{ we need this in the next step}\right) \qquad (5/2.1.1c,d)$$

Unknown integration constants C_1, C_2 can be obtained when applying suitable boundary conditions. But we do not know the displacements of the cylinder faces $\mathbf{u_1}$, $\mathbf{u_2}$. It follows from this that another quantity must be chosen for which we know the values at the faces. It is easy to find out that this role can be played only by radial stresses (σ_{r1} , σ_{r2}), because the faces are loaded with internal and external pressures $\mathbf{p_1}$, $\mathbf{p_2}$, respectively, acting in a radial direction (see Fig.7/2.1.1*a*).

6) Substituting 5) into 3b) ⇒ **stress expressions with unknown integral constants:**

$$\sigma_t = E^x \cdot \left[\frac{u}{x} + \mu \cdot u'\right] = E^x \cdot \left[C_1 + \frac{C_2}{x^2} + \mu C_1 - \mu \frac{C_2}{x^2}\right] = K + \frac{C}{x^2} \qquad \textbf{(6/2.1.1}\textit{a}\textbf{)}$$

$$\sigma_r = E^x \cdot \left[u' + \mu \cdot \frac{u}{x}\right] = E^x \cdot \left[C_1 - \frac{C_2}{x^2} + \mu C_1 + \mu \frac{C_2}{x^2}\right] = K - \frac{C}{x^2} \qquad \textbf{(6/2.1.1}\textit{b}\textbf{)}$$

Where $K = E^x \cdot C_1[1+\mu]$ and $C = E^x \cdot C_2[1-\mu]$ **(6/2.1.1*c,d*)**

are the integration constants in a new shape.

Note: After solving the integration constants *K, C* in step 7, we return to step 6 (then denoted 6a) to obtain final stress expressions.

7) **Boundary conditions:** σ_{r1}=-p_1 , σ_{r2}=-p_2 ⇒ expressions of the integral constants:

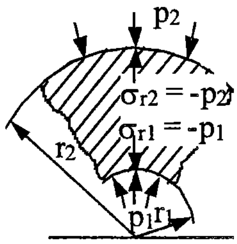


Fig. 7/2.1.1*a*

$$\left.\begin{aligned} \sigma_{r1} &= -p_1 = K - \frac{C}{r_1^2} \quad /\cdot(-r_1^2) \\ \sigma_{r2} &= -p_2 = K - \frac{C}{r_2^2} \quad /\cdot r_2^2 \end{aligned}\right\} + \Rightarrow K = \frac{p_1 \cdot r_1^2 - p_2 \cdot r_2^2}{r_2^2 - r_1^2} \qquad (7/2.1.1a)$$

$$\left.\begin{aligned} \sigma_{r1} &= -p_1 = K - \frac{C}{r_1^2} \quad /\cdot(-1) \\ \sigma_{r2} &= -p_2 = K - \frac{C}{r_2^2} \end{aligned}\right\} + \Rightarrow C = (p_1 - p_2)\frac{r_1^2 \cdot r_2^2}{r_2^2 - r_1^2} \qquad (7/2.1.1b)$$

Now, when applying the *second possible equilibrium equation* - **in the *axial direction*** - for closed cylinders - we learn that the integration constant *K* (Eq.7/2.1.1*a*) equals the <u>*axial* stress σ_a exerted in *closed* cylinders</u>:

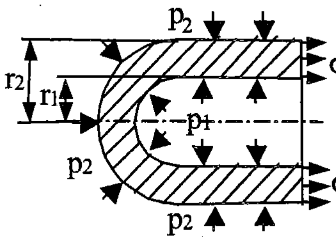


Fig.7/2.1.1*b*

$$\sigma_a \cdot (\pi \cdot r_2^2 - \pi \cdot r_1^2) - p_1 \cdot \pi \cdot r_1^2 + p_2 \cdot \pi \cdot r_2^2 = 0$$

$$\sigma_a = \frac{p_1 \cdot r_1^2 - p_2 \cdot r_2^2}{r_2^2 - r_1^2} \qquad (7/2.1.1c)$$

6a) Substituting 7) into 6) ⇒ **the final stress expressions:**

$$\sigma_t = K + \frac{C}{x^2} = \frac{p_1 \cdot r_1^2 - p_2 \cdot r_2^2}{r_2^2 - r_1^2} + (p_1 - p_2)\frac{r_1^2 \cdot r_2^2}{r_2^2 - r_1^2} \cdot \frac{1}{x^2} = (\sigma_a) + \frac{C}{x^2}$$

$$\sigma_r = K - \frac{C}{x^2} = \frac{p_1 \cdot r_1^2 - p_2 \cdot r_2^2}{r_2^2 - r_1^2} - (p_1 - p_2)\frac{r_1^2 \cdot r_2^2}{r_2^2 - r_1^2} \cdot \frac{1}{x^2} = (\sigma_a) - \frac{C}{x^2}$$

(6/2.1.1*e,f*)

Note: For practical applications, the last expressions of Eqs.**6**/2.1.1*e,f* are the most suitable (while having in mind the contents of symbols *K, or* (σ_a), and ***C***, Eqs.**7**/2.1.1*a,b*). According to these expressions we can see that *geometrically* the σ_t and σ_r stresses can be interpreted as *polytropic* curves which are **symmetric** with respect to their mutual asymptote (σ_a) (Fig.**8**/2.1.1*a*)

Since the constant *K* has the same shape as the axial stress (in closed cylinders) σ_a (compare Eqs. 7/2.1.1a,c) the symbol (σ_a) is used, while the parentheses express the fact that only the shape of these symbols coincides, but not always their interpretation:

1) the **mathematical** meaning of (σ_a) ...*an integration constant:* $(\sigma_a) = K$;
2) the **geometrical** meaning of (σ_a) ... *the location of the polytropic curve asymptote;*
3) the **physical** meaning of (σ_a) - only with **closed** cylinders... *the axial stress:* $(\sigma_a) = \sigma_a$.

Conclusion: Items **1-7 (6a)** are to be proceed for *deriving relations* applicable for all TC (***thick cylinder***) types.

2.1.2 Dimensioning and deformation of TC (steps 8 and 9)

The discussion of TC problems will continue with items needed when **computing concrete mechanical members made of thick cylinders.**

8) **Dimensioning:** a) single cylinders subjected to both internal and external overpressure

b) *pressed (compound) cylinders* subjected to *internal overpressure* – see 2.1.3.

a) **Single cylinder** **internal overpressure $p_1 > p_2$** **external overpressure $p_1 < p_2$**

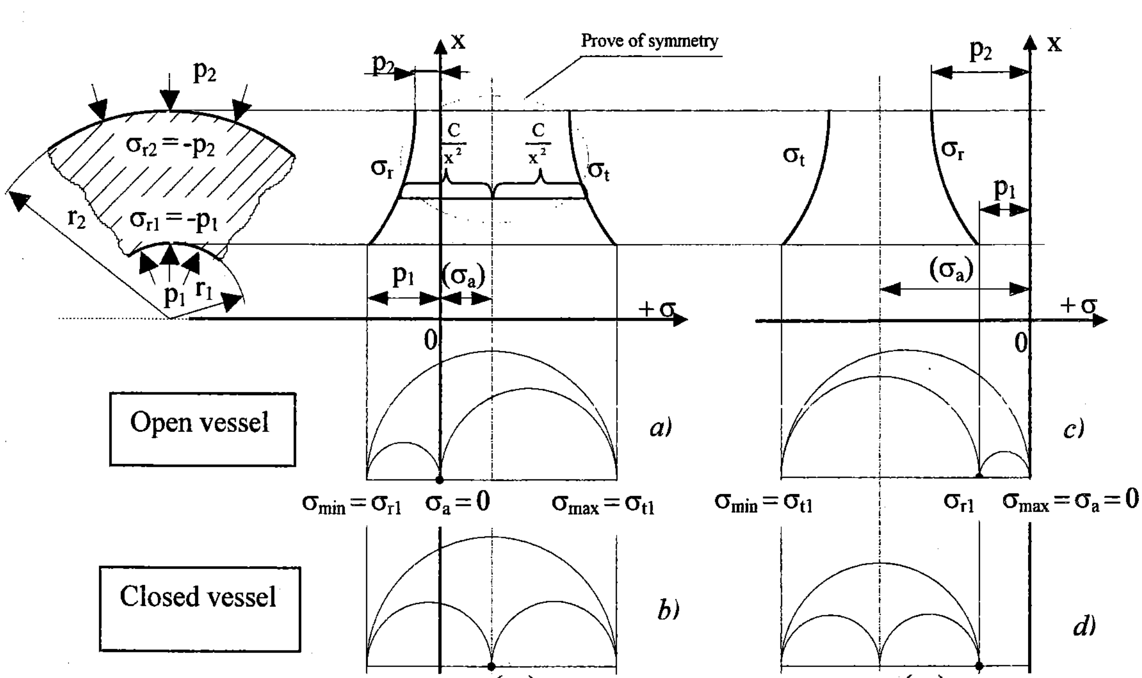


Fig.**8**/2.1.2*a,b,c,d* $\sigma_{min} = \sigma_{r1}$ $\sigma_a = (\sigma_a)$ $\sigma_{max} = \sigma_{t1}$ $\sigma_{min} = \sigma_{t1}$ $\sigma_a = (\sigma_a)$ $\sigma_{max} = \sigma_{r1}$

Strength criterion (Tresca):

Internal overpressure: $p_1 > p_2$

for **both open and closed** vessels

$$\sigma_{eq} = \sigma_{t1} - \sigma_{r1} \le \sigma_{all}$$

$$\sigma_{r1} = -p_1 \ ; \ \sigma_{t1} = 2(\sigma_a) + p_1$$

External overpressure: $p_1 < p_2$

for **open** vessels

$$\sigma_{eq} = \sigma_a - \sigma_{t1} \le \sigma_{all}$$

$$\sigma_a = 0 \ ; \ \sigma_{t1} = 2(\sigma_a) + p_1$$

$$(\sigma_a) = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2}$$

$$p_1 - p_2 \le \frac{\sigma_{all}}{2}\left[1 - \left(\frac{r_1}{r_2}\right)^2\right] \qquad (8/2.1.2.a)$$

$$(\sigma_a) = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2}$$

$$p_2 \le \frac{\sigma_{all}}{2}\left[1 - \left(\frac{r_1}{r_2}\right)^2\right] + \frac{p_1}{2}\left[1 + \left(\frac{r_1}{r_2}\right)^2\right] \qquad (8/2.1.2.b)$$

Note: When imagining that the cylinder thickness increases theoretically to infinity ($r_2 \to \infty$), it follows from (8/2.1.2.a) that the maximum borne internal overpressure of a single tube is strongly limited by the material and can attain as only as

$(p_1 - p_2)_{max} \to \sigma_{all}/2$.

In some cases, so-called **compound cylinders** can increase it – see 2.1.3.

for **closed** vessels

$$\sigma_{eq} = \sigma_{r1} - \sigma_{t1} \le \sigma_{all};\ \sigma_{r1} = -p_1;\ \sigma_{t1} = 2(\sigma_a) + p_1$$

$$(\sigma_a) = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2};\quad p_2 - p_1 \le \frac{\sigma_{all}}{2}\left[1 - \left(\frac{r_1}{r_2}\right)^2\right]$$

(8/2.1.2.c)

9) **Deformation, displacement**

Change in the radii of cylinders when loaded:

Since the displacement *u* means the change Δx in radius *x* (i.e., $u = \Delta x$), we apply the tangent strain formula Eq. 2.1.1.2*a*, where a suitable *generalized Hooke's law* is substituted:

Closed vessel (3D stress state): $\varepsilon_t = \frac{u}{x} = \frac{1}{E}[\sigma_t - \mu(\sigma_r + \sigma_a)]$;

open vessel (2D stress state, because $\sigma_a = 0$): $\varepsilon_t = \frac{u}{x} = \frac{1}{E}[\sigma_t - \mu\sigma_r]$ (9/2.1.2.*a,b*)

Change in the radii of the inner and outer vessel faces:

	Closed vessels:	Open vessels ($\sigma_\alpha = 0$):
At the inner face, $x = r_1$:	$\Delta r_1 = \frac{r_1}{E}[\sigma_{t1} - \mu(\sigma_a - p_1)]$	$\Delta r_1 = \frac{r_1}{E}[\sigma_{t1} + \mu \cdot p_1]$
At the outer face, $x = r_2$:	$\Delta r_2 = \frac{r_2}{E}[\sigma_{t2} - \mu(\sigma_a - p_2)]$	$\Delta r_2 = \frac{r_2}{E}[\sigma_{t2} + \mu \cdot p_2]$

(9/2.1.2.*c,d,e,f*)

Example: An open pressure vessel is loaded by external pressure $p_2 = 50\ MPa$.

Dimensions: $r_1 = 50\ mm$ and $r_2 = 80\ mm$; material: $\sigma_Y = 200\ MPa$, $E = 2.10^5\ MPa$, $\mu = 0.3$.

Determine: the vessel safety factor: k_Y; displacements of the vessel faces: Δr_1, Δr_2.

Solution: Open cylinder $\Rightarrow \sigma_a = 0$

Never forget to draw first the type of vessel stress distribution (here for **external overpressure**):

Position of asymptote: $(\sigma_a) = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2} = -p_2 \frac{r_2^2}{r_2^2 - r_1^2} = -50 \frac{80^2}{80^2 - 50^2} = -82.0\text{MPa}$

Stress state in the vessel faces:

$\sigma_{r1} = -p_1 = 0 \quad ; \quad \sigma_{t1} = 2(\sigma_a) + p_1 = 2(\sigma_a) = 2 \cdot (-82) = -164.0\text{MPa}$

$\sigma_{r2} = -p_2 = -50.0\text{MPa}$; $\sigma_{t2} = 2(\sigma_a) + p_2 = 2 \cdot (-82) + 50 = -114.0\text{MPa}$

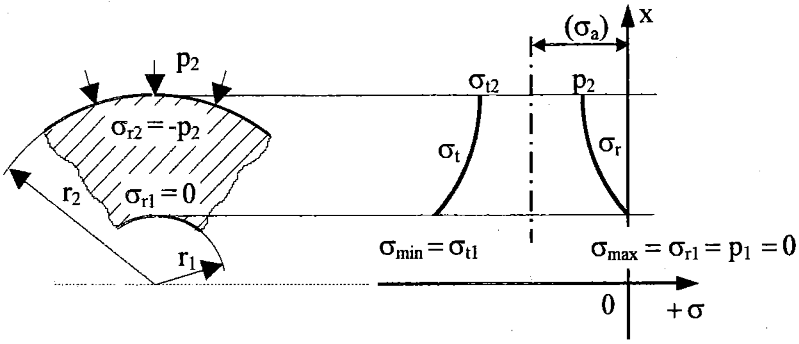
Tresca's strength criterion: $\sigma_{eq} = \sigma_{max} - \sigma_{min} = 0 - \sigma_{t1} = -\sigma_{t1} = 164.0\text{MPa}$

Factor of safety : $$k_Y = \frac{\sigma_Y}{\sigma_{eq}} = \frac{200}{164} = 1.22$$

Change in the face radii caused by the loading:

$$\Delta r_1 = \frac{r_1}{E}[\sigma_{t1} + \mu \cdot p_1] = \frac{50}{2 \cdot 10^5}(-164) = -0.0410 \text{ mm}$$

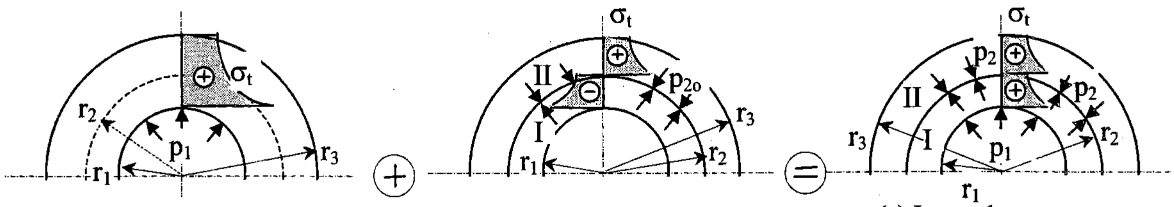
$$\Delta r_2 = \frac{r_2}{E}[\sigma_{t2} + \mu \cdot p_2] = \frac{80}{2 \cdot 10^5}(-114 + 0.3 \cdot 50) = -0.0396 \text{ mm}$$



2.1.3 Dimensioning and deformation of pressed (compound) cylinders (steps 8 & 9)

8) **Dimensioning:** pressed (compound) cylinders subjected to internal overpressure

From the sketch of the stress distributions in Fig.**8**/2.1.2*a,b* it is evident that there is a large variation in tangent (circumferential, or hoop) stress across the wall of a cylinder subjected to internal overpressure. The material of the cylinder is not therefore used to its best advantage. To obtain a more uniform hoop stress distribution, cylinders are often built up by shrinking one tube on the outside of another. When the outer tube ***II*** contracts on cooling the inner tube ***I*** is brought into a state of compression. The outer tube ***II*** will conversely be brought into a state of tension. If this compound cylinder ***I + II*** is now subjected to internal pressure p_1 the resultant hoop stresses will be the algebraic sum of those resulting from internal pressure and those resulting from shrinkage, as drawn in Fig.**8**/2.1.2*c*; thus a much smaller total fluctuation of hoop stress is obtained.



(α) Internal overpressure only (as if made of one piece)

(β) Shrinkage only

(γ) Internal overpressure - compound cylinder

Fig.**8**/2.1.3

9) **Deformation:** pressed (compound) cylinders **subjected to** internal overpressure

In order to achieve a required performance of the compound tubes, a corresponding shrinkage allowance has to be produced – see Fig.9/2.1.3.

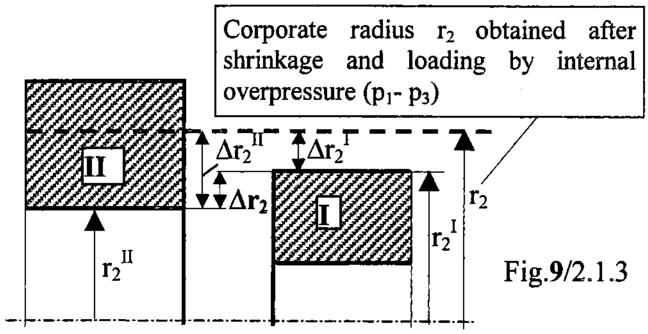


Fig.9/2.1.3

r_2^I, r_2^{II} ... before shrinkage

Δr_2 ... required shrinkage allowance

$$\Delta r_2 = \Delta r_2^{\,II} - \Delta r_2^{\,I} \,/\bullet \frac{1}{r_2}$$

$$(because \quad r_2^{\,II} \approx r_2^{\,I} \approx r_2)$$

$$\frac{\Delta r_2}{r_2} = \frac{\Delta r_2^{\,II}}{r_2^{\,II}} - \frac{\Delta r_2^{\,I}}{r_2^{\,I}} = \varepsilon_{t2}^{\,II} - \varepsilon_{t2}^{\,I} \Rightarrow$$

Applying generalized Hooke's law (while $\sigma_{r2}^{II} = \sigma_{r2}^{I} = -\,p_2$, see example below) we have

$$\Delta r_2 = \frac{r_2}{E}\left[\sigma_{t2}^{II} + \mu p_2 - \sigma_{t2}^{I} - \mu p_2\right] = \frac{r_2}{E}\left[\sigma_{t2}^{II} - \sigma_{t2}^{I}\right] = \frac{2r_2}{E}\left[(\sigma_a)^{II} - (\sigma_a)^{I}\right] \qquad (9/2.1.3a,b)$$

A procedure for computing *compound (pressed) cylinders* will be demonstrated when solving a concrete example:

Given: *Pressed cylinders: $r_1 : r_2 : r_3 = 2:5:9$, made of material: $\sigma_Y = 300$ MPa, are subjected to internal overpressure: $p_1 > p_3$.*

Tasks: 1) Assess allowable overpressure $(p_1 - p_3)_{all}$ - *apply step 8;*

2) *Determine necessary interference (shrinkage allowance)* $\boldsymbol{\Delta r_2}$ - apply step 9;

3) *Check the structure behaviour* (its strength) *at shrinkage state only* (when in the unloading state, i.e., $p_1 = 0$), see the case β in Fig.**8**/2.1.3.

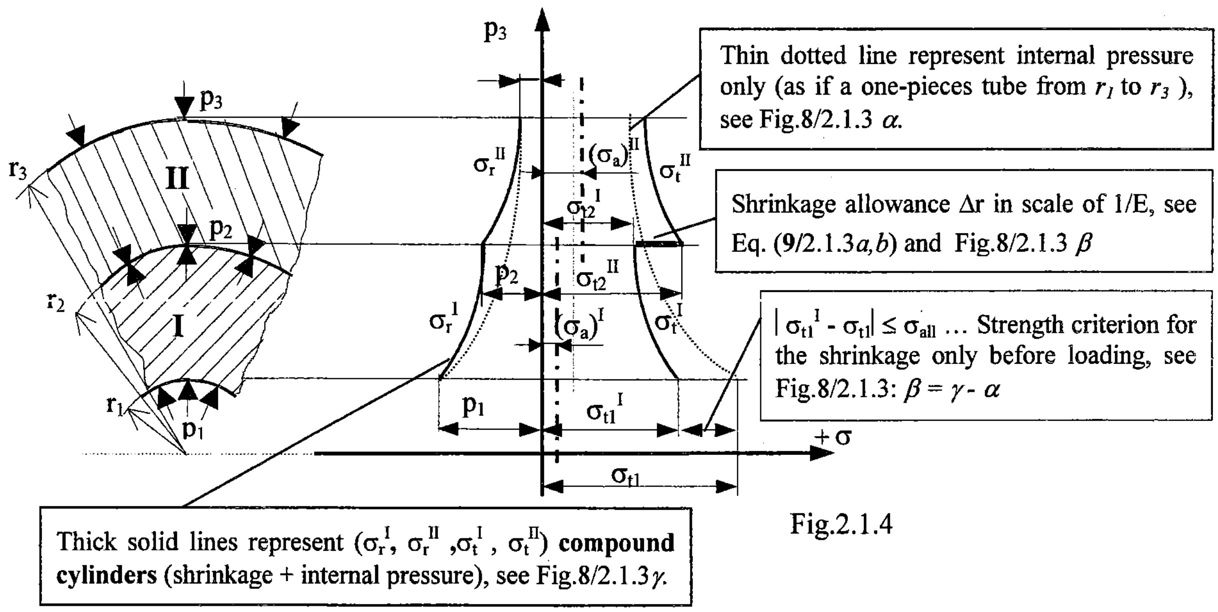


Fig.2.1.4

1) Allowable overpressure $(p_1 - p_3)_{all}$ - apply step 8

When considering internal overpressure exerted in the two pressed cylinders we can write **Strength criteria by Tresca ("τ_{max}")** successively:

II. (outer) cylinder (compare a single cylinder subjected to internal overpressure and change the respective subscripts compatibly)

$$\sigma_{eq}^{\,II} = \sigma_{t2}^{\,II} + p_2 = \left(2\sigma_a^{\,II} + p_2\right) + p_2 \le \sigma_{all} \;\Rightarrow p_2 - p_3 \le \frac{\sigma_{all}}{2}\left[1 - \left(\frac{r_2}{r_3}\right)^2\right] \qquad (1)$$

where $(\sigma_a)^{II} = \dfrac{p_2 r_2^{\,2} - p_3 r_3^{\,2}}{r_3^{\,2} - r_2^{\,2}}$

I. (inner) cylinder (compare a single cylinder subjected to internal overpressure)

$$\sigma_{eq}^{\ \ I} = \sigma_{t1}^{\ \ I} + p_1 = \left(2\sigma_a^{\ I} + p_1\right) + p_1 \le \sigma_{all} \Rightarrow p_1 - p_2 \le \frac{\sigma_{all}}{2}\left[1 - \left(\frac{r_1}{r_2}\right)^2\right] \qquad (2)$$

where $(\sigma_a)^I = \dfrac{p_1 r_1^{\,2} - p_2 r_2^{\,2}}{r_2^{\,2} - r_1^{\,2}}$ summing the above Eqs.(1) and (2). we obtain

$$p_1 - p_3 \le \frac{\sigma_{all}}{2}\left[2 - \left(\frac{r_1}{r_2}\right)^2 - \left(\frac{r_2}{r_3}\right)^2\right], \quad \Rightarrow \quad p_1 \le 229.7 \text{ MPa} \qquad (3)$$

Note: we introduce a concept of *optimum radius*: $r_{2,opt} = \sqrt{r_1 \cdot r_3} = 4.24$, an extreme of

$$p_1 - p_3 \le \frac{\sigma_{all}}{2}\left[2 - \left(\frac{r_1}{r_2}\right)^2 - \left(\frac{r_2}{r_3}\right)^2\right], \text{ when applying } \frac{\partial(p_1 - p_3)}{\partial r_2} = 0, \text{ leading to}$$

$$(p_1 - p_3)_{opt} \le \sigma_{all} \cdot \left[1 - \left(\frac{r_1}{r_3}\right)\right] = 233.\dot{3} \quad \text{MPa.}$$

2) <u>Necessary interference (shrinkage allowance) *Δr₂*</u> <u>(enabling proper operation of the structure) – step 9</u>

$$\Delta r_2 = \Delta r_2^{\ II} - \Delta r_2^{\ I} \ /\cdot \frac{1}{r_2} \Rightarrow$$

$$\frac{\Delta r_2}{r_2} \approx \frac{\Delta r_2^{\ II}}{r_2^{\ II}} - \frac{\Delta r_2^{\ I}}{r_2^{\ I}} = \varepsilon_{t2}^{\ \ II} - \varepsilon_{t2}^{\ \ I} = \frac{1}{E}\left[\left(\sigma_{t2}^{\ \ II} - \mu\sigma_{r2}^{\ \ II}\right) - \left(\sigma_{t2}^{\ \ I} - \mu\sigma_{r2}^{\ \ I}\right)\right]$$

and since it holds $\sigma_{r2}^{\ \ II} = \sigma_{r2}^{\ \ I} = -p_2$ we have

$$\Delta r_2 = \frac{r_2}{E}\left[\sigma_{t2}^{\ \ II} - \sigma_{t2}^{\ \ I}\right] = \frac{2 \cdot r_2}{E}\left[(\sigma_a)^{II} - (\sigma_a)^{I}\right] \qquad (4), (5)$$

Eqs. (4) and (5) hold for every type of pressed (two-layer) cylinder. But quantities $\sigma_{t2}^{\ II,I}$ and/or $(\sigma_a)^{ii,i}$ require $\mathbf{p_2}$ to be calculated, which we did not need for the resulting strength criterion Eq.(3).

Assuming equality in Eqs.(1), (2), we have

$$p_2 = 103.7 \text{ MPa.}, \ (\sigma_a)^{II} = 46.3 \text{ MPa, and } (\sigma_a)^{I} = -79.7 \text{ MPa.}$$

After substituting, Eq.(5) yields....... $\Delta r_2 = 0.756$ mm.

From the equalities in Eqs.(1) and (2), we can also obtain expressions

$(\sigma_a)^{II} = \dfrac{\sigma_{all}}{2} - p_2$ and $(\sigma_a)^{I} = \dfrac{\sigma_{all}}{2} - p_1$ leading to new, simpler, shapes of Eq.(5):

$$\Delta r_2 = \frac{2 \cdot r_2}{E}\left[(\sigma_a)^{II} - (\sigma_a)^{I}\right] = \frac{2 \cdot r_2}{E}\left(p_1 - p_2\right) = \frac{\sigma_{all} \cdot r_2}{E}\left[1 - \left(\frac{r_1^{\,2}}{r_2^{\,2}}\right)\right] \qquad (5), (6), (7)$$

<u>Expressions (6) and (7) **can be applied only when a pressed two-layer cylinder is designed**</u> based on the strength criteria (when equalities in Eqs.(1), (2), (3) can be assumed).

Note: The shape of Eq.(7) was obtained when the equality in Eq.(2) was substituted into Eq.(6).
You can readily prove that all these equations yield **Δr_2 = 0.756 mm.**

3) <u>Checking the strength of the structure when in unloading state, i.e., $p_1 = 0$</u>

Imagine that a compound vessel is manufactured by pressing one on the other, i.e., the outer (II) cylinder can be heated and put on the inner (I) cylinder. After cooling, cylinder II will load by shrinkage this whole compound structure. The previous strength calculations concern the structure when operating by internal overpressure $p_1 - p_3$. The unloading state ($p_1 = 0$) of this completed structure needs to be checked, since we cannot know in advance if the pressed cylinders are not damaged in a different stress condition ($p_1 = 0$). This procedure can be performed by superposing different sets of stress states of the structure based on different loading conditions, which we can see in Fig.**8**/2.1.3β. When subtracting a fictitious stress state (assuming a fictitious single, non-parted, i.e., made of a single piece of material, cylinder of radii r_1 and r_3), Fig.**8**/2.1.3α, from the stress state of the pressed cylinder, Fig.**8**/2.13γ, composed of two pieces, both under operating conditions, i.e., subjected to pressure p_1–p_3, we obtain the looked-for stress state of an unloaded pressed cylinder:

$$\left|\sigma^{I}_{t1} - \sigma_{t1}\right| \le \sigma_{all} \Rightarrow 183.2 \le 300 \text{ [MPa]}, \Rightarrow \underline{\text{the compound cylinder } \textbf{satisfies} \text{ all requirements}}$$

Where ... $\left|\sigma^{I}_{t1} = 2(\sigma_a)^{I}\right| + p_1 = 70.3\,\text{MPa}$ is the *tangent stress* exerted on the ***compound structure*** *inner face* caused by pressure p_1–p_3

$\left|\sigma_{t1} = 2(\sigma_a)\right| + p_1 = 253.5\,\text{MPa}$ is the *tangent stress* exerted on the ***fictitious*** *one-piece structure inner face* caused by pressure p_1–p_3

2.2 Circular plates

2.2.1 <u>Plate elastic stress formulas</u>

Assumptions for the CP solution:

1/ CP ***thickness t*** is relatively small in comparison with its ***radius r***, i.e., ***t << r***

2/ CP obeys the so-called *Kirchhoff hypothesis, which states*: CP cylindrical sections change into conical sections after deformation, i.e., *CP is stressed in bending* with its middle surface remaining non-stressed.

3/ CP deflection is small in comparison with the thickness. (If the deflection had exceeded half of the CP thickness then stretching of the CP middle surface would have been taken into consideration).

To derive elastic formulae of circular (Kirchhoff's) plates (CP) we can apply items similar to those for thick cylinders – see section 2.1.

1/ **Differential equilibrium equation** for circular plates (DEECP)

Based on Fig.**1**/2.2.1*a,b,c,d* the following formulae are obtained:

The *shearing forces* exerting on the element at a cylindrical section of radius *x* are

$$dV_{(x)} = \frac{Q(x)}{2\pi x} x d\alpha = \frac{Q(x)}{2\pi} d\alpha, \quad dV^{*}(x + dx) = \frac{Q(x) + dQ}{2\pi} d\alpha \qquad (\mathbf{1}/2.2.1a,b)$$

where *Q(x)* represents the transversal loading of the plate centre up to radius *x*.

After neglecting the infinitesimal quantities of higher order, we express the moment produced by the shearing forces in the following form

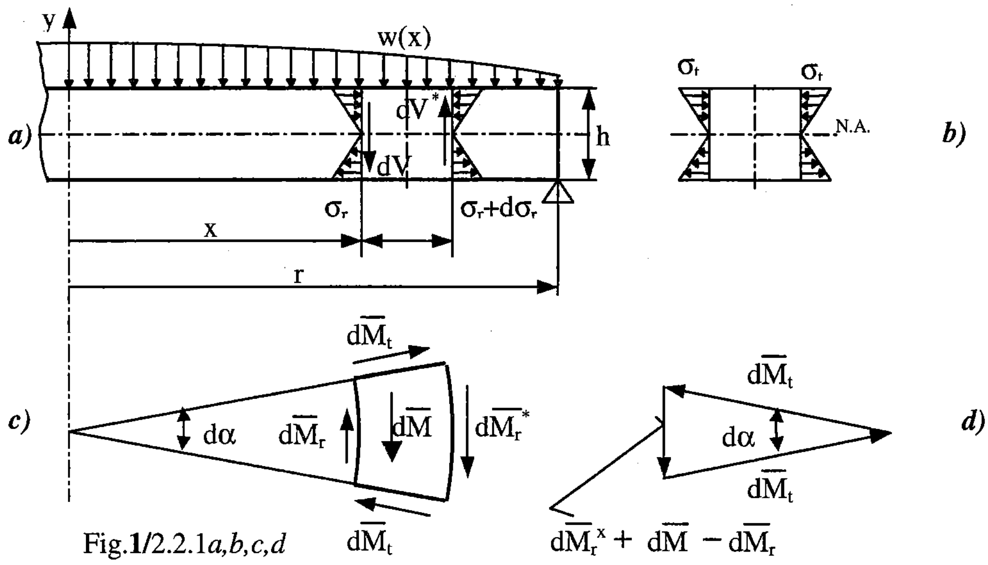


Fig.1/2.2.1*a,b,c,d*

$$dM = dV \cdot dx = \frac{Q(x)}{2\pi} d\alpha \cdot dx \qquad (1/2.2.1c)$$

The tangential and radial moments as resultants of tangent σ_t and radial σ_r stresses, respectively, at radius x :

$$dM_t = Z_{bt} \cdot \sigma_t = \frac{1}{6} \cdot dx \cdot h^2 \cdot \sigma_t \quad \text{and} \quad dM_r = Z_{br} \cdot \sigma_r = \frac{1}{6} \cdot x \cdot d\alpha \cdot h^2 \cdot \sigma_r \qquad (1/2.2.1d,e)$$

The radial moment at the radius $(x + dx)$ as a resultant of the radial stress $(\sigma_r + d\sigma_r)$

$$dM_r^* = Z_{br}^* \cdot \left(\sigma_r + d\sigma_r\right) = \frac{1}{6} \cdot (x + dx) \cdot d\alpha \cdot h^2 \cdot \left(\sigma_r + d\sigma_r\right) \qquad (1/2.2.1f)$$

After constructing the equilibrium triangle (Fig.1/2.2.1*d*), we can write the equilibrium equation successively using the following expressions:

$$dM_r^* + dM - dM_r = dM_t d\alpha \Rightarrow$$

$$d(\sigma_r x) - \sigma_t dx = -\frac{6Q(x)}{2\pi h^2} dx \Rightarrow \frac{d\sigma_r}{dx} x + \sigma_r - \sigma_t = -\frac{6Q(x)}{2\pi h^2} \qquad (1/2.2.1g,h)$$

2/ **Compatibility relations** (relations between strains and displacements)

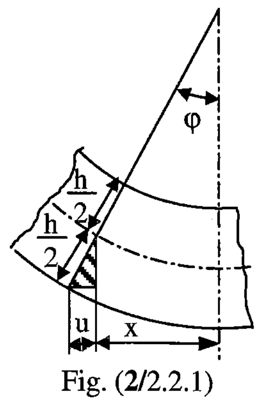


Fig. (2/2.2.1)

When dealing with circular plates, it is more convenient to express the radial displacement u of the bottom face by means of the slope φ of the plate middle (neutral) surface, i.e.

$$u = \frac{h}{2} \cdot \varphi, \qquad (2/2.2.1a)$$

cf. Fig. (2/2.2.1) and the respective strains can be expressed

$$\varepsilon_t = \frac{u}{x} = \frac{h}{2}\frac{\varphi}{x}, \quad \varepsilon_r = \frac{du}{dx} = \frac{h}{2}\frac{d\varphi}{dx} \qquad (2/2.2.1b,c)$$

3/ Generalized Hooke's Law

As it was done with thick cylinders, we express the generalized Hooke's law in three modes:

a/ see Eq.**3**/2.1.1*a* and then, after substituting $u = \frac{h}{2} \cdot \varphi$ and $u' = \frac{h}{2} \cdot \varphi'$, we have:

b/ $$\sigma_t = \frac{E}{1-\mu^2}\left[\varepsilon_t + \mu\varepsilon_r\right] = E^{\times}\frac{h}{2}\left[\frac{\varphi}{x} + \mu\varphi'\right] \qquad \textbf{(3/2.2.1}a\textbf{)}$$

$$\sigma_r = \frac{E}{1-\mu^2}\left[\varepsilon_r + \mu\varepsilon_t\right] = E^{\times}\frac{h}{2}\left[\varphi' + \mu\frac{\varphi}{x}\right] \qquad \textbf{(3/2.2.1}b\textbf{)}$$

and, finally, we also differentiate the relation for the radial stress which we need to substitute into the first step

c/ $$\frac{d\sigma_r}{dx} = E^{\times}\frac{h}{2}\left[\varphi'' + \mu\frac{\varphi' x - \varphi}{x^2}\right] \qquad \textbf{(3/2.2.1}c\textbf{)}$$

4/ **Basic circular plate differential equation** (BCPDE)

Substituting Eqs.**3**/2.2.1*b,c* into Eq.**1**/2.2.1*h*) we obtain

$$E^{\times}\frac{h}{2}\left[\varphi'' + \mu\frac{\varphi' x - \varphi}{x^2}\right]x + E^{\times}\frac{h}{2}\left[\varphi' + \mu\frac{\varphi}{x}\right] - E^{\times}\frac{h}{2}\left[\frac{\varphi}{x} + \mu\varphi'\right] = -\frac{6Q(x)}{2\pi h^2} \Big/ \cdot \frac{2}{E^{\times}h}$$

And finally we have $$\varphi'' x + \varphi' - \frac{\varphi}{x} = -\frac{6Q(x)}{E^{\times}\pi \cdot h^3} \qquad \textbf{(4/2.2.1}a\textbf{)}$$

We can see that BCPDE is a non-homogeneous differential equation with the left hand side having the same form as in the case of a thick cylinder. While for BTCDE, which being a homogeneous differential equation, we were to find only a general solution, for BCPDE, we are to obtain, into addition, a necessary particular integral depending on the right hand side given by the type of plate transversal loading.

This task can be simplified by rewriting the left hand side into the following shape

$$x\left[\frac{1}{x}(\varphi \cdot x)'\right]' = -\frac{6Q(x)}{E^{\times}\pi \cdot h^3} \qquad \textbf{(4/2.2.1}b\textbf{)}$$

since this differential eq. can be solved by applying a simple integration twice.

Note: Since each CP is loaded by a generally different type of load *Q(x)*, steps 5 – 9 will be solved for concrete practical examples (thus differing from TC, where concrete examples are applied using steps 8 and 9)

9/ **Deformation of plates** (theoretical approach)

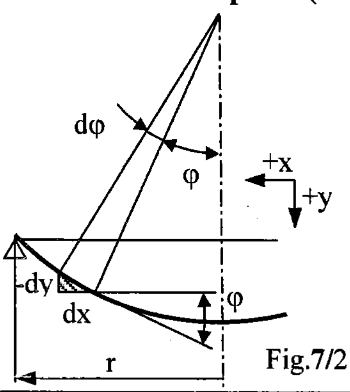


Fig.7/2.2.1

Plate deformation is presented by the change of its neutral, i.e., middle, originally plane, surface into a surface bent in the form of a rotationally symmetric cup. The equation of this surface can be derived from the deformation of a plate element taken in an axial section, Fig.7/2.2.1:

$$dy = -\varphi \cdot dx\ ,$$

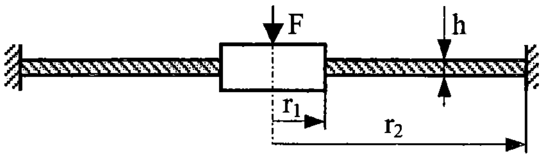
where the negative increment *dy* corresponds with positive *dx.*

The resulting plate deflection can be calculated from

$$y = K - \int \varphi \cdot dx \qquad (7/2.2.1a)$$

where the *integration constant K* can be obtained by applying a corresponding *boundary condition* (BC). For instance, BC for the plate in Fig.7/2.2.1 is $y_{(x=r)} = 0$.

Example 1:



Parameters given:
i) geometry: *t = 10 mm; r_1 = 100 mm; r_2 = 1000 mm*

ii) material: *σ_{all} = 100 MPa*
E = 2.10⁵ MPa → *$E = 2.10^5$ MPa*
μ = 0.3

Task: allowable load ***F***

We start from item **4** (the basic circular plate differential equation - BCPDE):

$x \cdot \left[\frac{1}{x} \cdot (\varphi \cdot x)'\right]' = -\frac{6 \cdot Q(x)}{\pi \cdot E^x \cdot h^3}$, where the force ***Q(x) = F*** (acting on the plate area within radius *x*)

will be substituted. After denoting $B = \frac{6 \cdot F}{\pi \cdot E^x \cdot h^3}$, we have the following diff. eq.:

$$x \cdot \left[\frac{1}{x} \cdot (\varphi \cdot x)'\right]' = -B \ .$$

Item 5 (BCPDE can be solved by applying "per partes" integration):

$$\varphi = -\frac{B}{2} \cdot x \cdot \ln x + \underbrace{\frac{B}{4} \cdot x + \frac{C_1'}{2} \cdot x}_{C_1 \cdot x} + \frac{C_2}{x} \Rightarrow \varphi = C_1 \cdot x + \frac{C_2}{x} - \frac{B}{2} \cdot x \cdot \ln x$$

Item 7 (item 6 is omitted, since we know BC for φ):

$$\text{B.C.} \quad \left.\begin{array}{l} \varphi(r_1) = 0 \\ \varphi(r_2) = 0 \end{array}\right\} \Rightarrow C_1 \cdot r_1 + \frac{C_2}{r_1} - \frac{B}{2} \cdot r_1 \cdot \ln r_1 = 0; \quad C_1 \cdot r_2 + \frac{C_2}{r_2} - \frac{B}{2} \cdot r_2 \cdot \ln r_2 = 0$$

which yields:
$$C_1 = \frac{B}{2} \cdot \frac{r_2^2 \cdot \ln r_2 - r_1^2 \cdot \ln r_1}{r_2^2 - r_1^2} > 0; \quad C_2 = -\frac{B}{2} \cdot \frac{\ln r_2 - \ln r_1}{r_2^2 - r_1^2} \cdot r_1^2 \cdot r_2^2 < 0$$

Item 6 (applying GHL we express stresses):

The tangent stress:

$$\sigma_t = \frac{E^x \cdot h}{2} \cdot \left[\frac{\varphi}{x} + \mu \cdot \varphi'\right]; \sigma_t = \frac{E^x \cdot h}{2} \cdot \left[C_1 + \frac{C_2}{x^2} - \frac{B}{2} \cdot \ln x + \mu \cdot \left(C_1 - \frac{C_2}{x^2} - \frac{B}{2} \cdot \ln x - \frac{B}{2}\right)\right]$$

$\sigma_t = C_1^* + \frac{C_2^*}{x^2} - B^* \cdot \ln x$, where we expressed new types of integration constants

$$C_1^* = \frac{E^x \cdot h}{2} \cdot \left[C_1 \cdot (1+\mu) - \frac{B}{2} \cdot \mu\right]; \quad C_2^* = \frac{E^x \cdot h}{2} \cdot C_2 \cdot (1-\mu); \quad B^* = \frac{E^x \cdot h}{4} \cdot B \cdot (1+\mu)$$

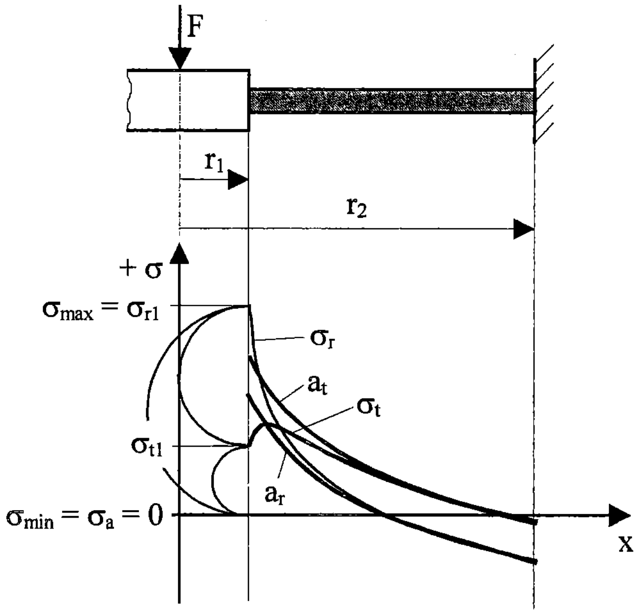
to enable a better survey of the stress creating functions, which are:

the *asymptotic curve*... $a_t = C_1^* - B^* \cdot \ln x$; the *polytropic curve* of the second order... $\frac{C_2}{x^2}$.

The radial stress

$$\sigma_r = \frac{E^x \cdot h}{2} \cdot \left[\varphi' + \mu \cdot \frac{\varphi}{x} \right] \Rightarrow \sigma_r = C_1^{**} - \frac{C_2^*}{x^2} - B^* \cdot \ln x$$

(where... $C_1^{**} = \frac{E^x \cdot h}{2} \cdot \left[C_1 \cdot (1+\mu) - \frac{B}{2} \right]$ and the asymptotic line ... $a_r = C_1^{**} - B^* \cdot \ln x$)



Item 8: Tresca's Strength criterion

$$\sigma_{eq} = \sigma_{max} - \sigma_{min} \le \sigma_{all}$$

$$\sigma_r(r_1) - \sigma_a \le \sigma_{all}, \quad (\sigma_a = 0)$$

$$\underline{\sigma_r(r_1) \le \sigma_{all}} \Rightarrow$$

$$C_1^{**} - \frac{C_2^*}{r_1^2} - B^* \cdot \ln r_1 \le \sigma_{all}$$

$$\frac{B}{2} \cdot A \le \frac{2 \cdot \sigma_{all}}{E^x \cdot h}$$

$$\frac{1}{2} \cdot \frac{6 \cdot F}{\pi \cdot E^x \cdot h^3} \cdot A \le \frac{2 \cdot \sigma_{all}}{E^x \cdot h},$$

$$F \le \frac{2 \cdot \pi \cdot h^2 \cdot \sigma_{all}}{3 \cdot A}; \quad F \le 5735.4\text{N}$$

(where... $A = \frac{r_2^2 \cdot \ln r_2 - r_1^2 \cdot \ln r_1}{r_2^2 - r_1^2} \cdot (1+\mu) + \frac{\ln r_2 - \ln r_1}{r_2^2 - r_1^2} \cdot r_1^2 \cdot r_2^2 \cdot (1-\mu) - (1+\mu) \cdot \ln r_1 - 1$)

Item 8: Deflection of the plate ($y = K - \int \varphi \cdot dx$)

$$y = K - \int \left(C_1 \cdot x + \frac{C_2}{x} - \frac{B}{2} \cdot x \cdot \ln x \right) dx \Rightarrow y = K - \left[\frac{C_1}{2} \cdot x^2 + C_2 \cdot \ln x - \frac{B}{2} \cdot \left(\frac{x^2}{2} \cdot \ln x - \frac{x^2}{4} \right) \right]$$

B.C. for deflection:

$$y(r_2) = 0 = K - \left[\frac{C_1}{2} \cdot r_2^2 + C_2 \cdot \ln r_2 - \frac{B}{2} \cdot \left(\frac{r_2^2}{2} \cdot \ln r_2 - \frac{r_2^2}{4} \right) \right] \Rightarrow K = \frac{3 \cdot F}{\pi \cdot E^x \cdot h^3} \cdot D = 2.24$$

(where $D = \frac{r_2^2 \cdot \ln r_2 - r_1^2 \cdot \ln r_1}{2 \cdot (r_2^2 - r_1^2)} \cdot r_2^2 - \frac{\ln r_2 - \ln r_1}{r_2^2 - r_1^2} \cdot r_1^2 \cdot r_2^2 \cdot \ln r_2 + \left(\frac{r_2^2}{2} \cdot \ln r_2 - \frac{r_2^2}{4} \right)$)

Maximum deflection: $y_{max} = y(r_1) = 4.3\text{mm}$

2.2.2 More general types of CP loading

Example 2:

In **example 1**, we see how to proceed when a CP is simply loaded, i.e., when a single BCPDE is sufficient for solving it. Now, a more complex case will be solved demanding several BCPDEs to be applied, Fig.2.2.2.1.

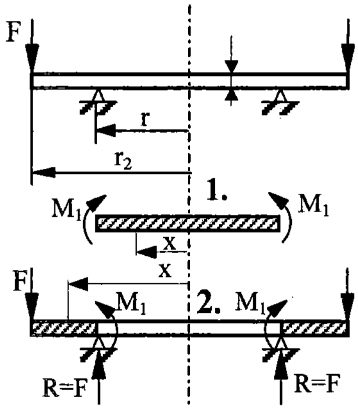


Fig.2.2.2.1

The plate, being simply supported by a ring support with radius r_1, is subjected to ring load F along its outer circumference (radius r_2).

Item 4: (substitution in Eq. 4/2.2.1*b*):

Plate **I**: $Q(x) = 0$; Plate **II**: $Q(x) = -F$

Item 6 (GHL) and *7* (BC):

$x = 0 : \varphi^{I}(0) = 0$; (BC… 1)

$$x = r_1:\ \sigma^{I}(r_1) = \sigma^{II}(r_1) = \frac{M_1}{2\pi r_1}\cdot\frac{6}{h^2} \quad \text{(BC… 2)}$$

$$\varphi^{I}(r_1) = \varphi^{II}(r_1) \quad \text{(BC… 3)}$$

$x = r_1$: $\varphi^{II}(r_2) = 0$ (BC… 4)

3. 3D - stress and strain state, differential equilibrium equations of continuum elements

3.1 General stress state

Stress is a symmetric tensor of the second order:

$$\sigma_{ij} = \begin{Bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{Bmatrix} = \begin{Bmatrix} \sigma_x & \tau_z & \tau_y \\ \tau_z & \sigma_y & \tau_x \\ \tau_y & \tau_x & \sigma_z \end{Bmatrix}$$

3.1.1 First task: normal, shearing and resulting stresses on an oblique ρ plane

From the viewpoint of the general theory of tensors, the formal and at the same time basic feature of the tensor character of a state of stress at a given point is that, as one passes from the coordinate planes (*Oxy, Oyz, Ozx*) to an arbitrary inclined plane ρ with outward normal ***n*** (making angles α, β, γ with the respective coordinates *x, y, z*), the stress components ν_x , ν_y , ν_z of the total stress ν (∢ φ, ψ, χ with the respective coordinates *x, y, z*), which is exerted on plane ρ, are expressed by:

$$\begin{aligned} \nu_x &= \sigma_x \cdot \cos\alpha + \tau_z \cdot \cos\beta + \tau_y \cdot \cos\gamma \\ \nu_y &= \tau_z \cdot \cos\alpha + \sigma_y \cdot \cos\beta + \tau_x \cdot \cos\gamma \\ \nu_z &= \tau_y \cdot \cos\alpha + \tau_x \cdot \cos\beta + \sigma_z \cdot \cos\gamma \end{aligned} \qquad (3.1.1.1a,b,c)$$

These formulas are linear with respect to the original components as well as to the direction cosines.

Note: These equations is easy to obtain when expressing the equilibrium equation of forces acting in a respective coordinate direction of a cut corner of a stress cube (i.e., "the stress at a point tetrahedron") - see Fig. 3.1.1.1. For instance, in the *x* direction:

$$\begin{aligned} &\nu_x \cdot dA - \sigma_x \cdot dA_x - \tau_y \cdot dA_z - \tau_z \cdot dA_y = \\ &= \nu_x \cdot dA - \sigma_x \cdot dA \cdot \cos\alpha - \tau_y \cdot dA \cdot \cos\gamma - \\ &- \tau_z \cdot dA \cdot \cos\beta = 0 \Rightarrow \\ &\nu_x = \sigma_x \cdot \cos\alpha + \tau_z \cdot \cos\beta + \tau_y \cdot \cos\gamma \end{aligned}$$

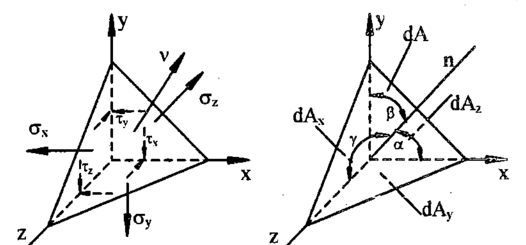


Fig. 3.1.1.1

Under a complete transformation involving the passage from axes *x, y, z* to new axes *x', y', z'* (one axis - say *x'*- of which coincides with the normal *n* of the above mentioned arbitrary inclined plane ρ, i.e. ***x'*** ≡ *n*) the components of the state of stress are expressed in terms of the original components $\sigma_x, \sigma_y, \sigma_z, \tau_x, \tau_y, \tau_z$ by projecting the components ν_x , ν_y , ν_z of the total strain ν, which is exerted in the plane ρ, on the *x'*≡*n* axis (according to Fig.), thus obtaining *normal (direct) stress*:

$$\begin{aligned} \sigma_\rho &= \sigma_{x'} = \nu_x \cdot \cos\alpha + \nu_y \cdot \cos\beta + \nu_z \cdot \cos\gamma = \\ &= \sigma_x \cdot \cos^2\alpha + \sigma_y \cdot \cos^2\beta + \sigma_z \cdot \cos^2\gamma + \\ &+ 2 \cdot \tau_x \cdot \cos\beta \cdot \cos\gamma + 2 \cdot \tau_y \cdot \cos\alpha \cdot \cos\gamma + 2 \cdot \tau_z \cdot \cos\alpha \cdot \cos\beta \end{aligned} \qquad (3.1.1.2)$$

This expression is linear with respect to the original components and quadratic (or so-called bilinear) with respect to the direction cosines of the new system.

The other two stress components exerted in this arbitrary inclined plane ρ are *shearing stresses* $\tau_{x'}$, $\tau_{z'}$ that can be obtained:

a/ <u>either</u> by projecting the components ν_x , ν_y , ν_z of the total strain ν, which is exerted in plane ρ, on the *y'* and *z'* axes, respectively (these axes must lie in plane ρ and form a rectangular coordinate system with the $x'\equiv n$ axis);

b/ <u>or</u> by projecting the total τ_ρ exerted in plane ρ on the *y'* and *z'* axes. The shearing strain τ_ρ is obtained by applying Pythagoras' theorem:

$$\tau_\rho = \sqrt{\nu_\rho^{\ 2} - \sigma_\rho^{\ 2}} = \sqrt{\nu_x^{\ 2} + \nu_y^{\ 2} + \nu_z^{\ 2} - \sigma_\rho^{\ 2}} =$$
$$= \sqrt{\sigma_1^{\ 2}\cdot\cos^2\alpha + \sigma_2^{\ 2}\cdot\cos^2\beta + \sigma_3^{\ 2}\cdot\cos^2\gamma - \left(\sigma_1\cdot\cos^2\alpha + \sigma_2\cdot\cos^2\beta + \sigma_3\cdot\cos^2\gamma\right)^2} \qquad (3.1.1.3)$$

3.1.2 <u>Second task: Principal stresses and principal planes (stress invariants; Eigen values and Eigen vectors)</u>

Consider the special case of the "stress at a point tetrahedron" where plane ρ is a principal plane subjected to a principal stress σ and, by definition, zero shear stress. The normal stress σ is thus coincident with the resultant stress ν, i.e., ν = σ, while both have the same direction cosines: $\cos\alpha = \cos\varphi$; $\cos\beta = \cos\psi$; $\cos\gamma = \cos\chi$.

Then

$$\nu_x = \nu\cdot\cos\varphi = \sigma\cdot\cos\alpha$$
$$\nu_y = \nu\cdot\cos\psi = \sigma\cdot\cos\beta \qquad (3.1.2.1a,b,c)$$
$$\nu_z = \nu\cdot\cos\chi = \sigma\cdot\cos\gamma$$

i.e., substituting in Eqs.3.1.1.1 we have

$$\sigma\cdot\cos\alpha = \sigma_x\cdot\cos\alpha + \tau_z\cdot\cos\beta + \tau_y\cdot\cos\gamma$$
$$\sigma\cdot\cos\beta = \tau_z\cdot\cos\alpha + \sigma_y\cdot\cos\beta + \tau_x\cdot\cos\gamma \qquad (3.1.2.2a,b,c)$$
$$\sigma\cdot\cos\gamma = \tau_y\cdot\cos\alpha + \tau_x\cdot\cos\beta + \sigma_z\cdot\cos\gamma$$

and after rearranging we have

$$0 = (\sigma_x - \sigma)\cdot\cos\alpha + \tau_z\cdot\cos\beta + \tau_y\cdot\cos\gamma$$
$$0 = \tau_z\cdot\cos\alpha + (\sigma_y - \sigma)\cdot\cos\beta + \tau_x\cdot\cos\gamma \qquad (3.1.2.3a,b,c)$$
$$0 = \tau_y\cdot\cos\alpha + \tau_x\cdot\cos\beta + (\sigma_z - \sigma)\cdot\cos\gamma$$

Considering Eqs. 3.1.2.3*a,b,c* as a set of three homogeneous linear equations in unknowns $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, the direction cosines of the principal plane, one possible solution, viz. $\cos\alpha = \cos\beta = \cos\gamma = 0$, can be dismissed since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ must always be maintained. The only other solution which gives real values for the direction cosines is that obtained by equating the determinant of the R.H.S. to zero: i.e.,

$$\begin{vmatrix} \sigma_x - \sigma & \tau_z & \tau_y \\ \tau_z & \sigma_y - \sigma & \tau_x \\ \tau_y & \tau_x & \sigma_z - \sigma \end{vmatrix} = 0 \qquad (3.1.2.4)$$

Evaluating the determinant yields the so-called "characteristic equation"

$$\sigma^3-(\sigma_x+\sigma_y+\sigma_z)\cdot\sigma^2+\left(\begin{vmatrix}\sigma_y & \tau_x\\ \tau_x & \sigma_z\end{vmatrix}+\begin{vmatrix}\sigma_z & \tau_y\\ \tau_y & \sigma_x\end{vmatrix}+\begin{vmatrix}\sigma_x & \tau_z\\ \tau_z & \sigma_y\end{vmatrix}\right)\cdot\sigma-$$

$$-\begin{vmatrix}\sigma_x & \tau_z & \tau_y\\ \tau_z & \sigma_y & \tau_x\\ \tau_y & \tau_x & \sigma_z\end{vmatrix}=0 \qquad \Rightarrow \qquad \sigma^3-I_1\cdot\sigma^2+I_2\cdot\sigma-I_3=0 \tag{3.1.2.5}$$

Thus, for any given set of Cartesian stress components in three dimensions a solution of this cubic equation is required before the principal stress value can be determined; a graphical solution is not possible.

Note: The solution for the principal stresses σ_1, σ_2 and σ_3 from the characteristic equation are known as the **Eigen values**, whilst the associated direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are termed the **Eigen vectors.**

Stress invariants

If, for the same applied stress system, the stress components had been given relative to some other set of Cartesian co-ordinates x', y' and z', the above equation would still apply (with x' replacing x, y' replacing y and z' replacing z) and would still produce the same principal stress values. It follows, therefore, that whatever axis system is chosen the coefficients of the various terms of the characteristics equation must have the same values, i.e., they are "non-varying quantities" or "invariants", and they are denoted I_1, I_2, I_3 in Eq.3.1.2.5. When evaluating the determinants applied in Eq.3.1.2.5 we can find expressions for all the invariants in the following shapes:

$$I_1=\sigma_x+\sigma_y+\sigma_z$$
$$I_2=(\tau_z^2+\tau_x^2+\tau_y^2)-(\sigma_x\cdot\sigma_y+\sigma_y\cdot\sigma_z+\sigma_z\cdot\sigma_x) \tag{3.1.2.6a}$$
$$I_3=\sigma_x\cdot\sigma_y\cdot\sigma_z+2\cdot\tau_x\cdot\tau_y\cdot\tau_z-\sigma_x\cdot\tau^2{}_x-\sigma_y\cdot\tau^2{}_y-\sigma_z\cdot\tau^2{}_z$$

Note: If the reference axes selected are principal stress axes in the system then all shear components reduce to zero and Eqs. (3.1.7a) reduce to:

$$I_1=\sigma_1+\sigma_2+\sigma_3;\; I_2=-(\sigma_1\cdot\sigma_2+\sigma_2\cdot\sigma_3+\sigma_3\cdot\sigma_1);\; I_3=\sigma_1\cdot\sigma_2\cdot\sigma_3 \tag{3.1.2.6b}$$

The first and second invariants are particularly important in the development of the theory of plasticity, since it is assumed that:

i) I_1 has no influence on initial yielding

ii) I_2 (= constant) can be taken as an important criterion of yielding

[Recall the *energy strength criterion* (based on the Maximum-shear-strain-energy, or distortion-energy, criterion), which is called the *HMH Criterion* (by its authors: Huber, Mises, Hencky)]

Alternative procedure for determination of principal stresses (eigen values)

An alternative solution to the characteristic cubic equation expressed in stress invariant format, viz. Eq. 3.1.2.5, is as follows:

principal stresses obtained:

$$\sigma_1=2S\cdot\cos(\alpha/3)+I_1/3$$
$$\sigma_2=2S\cdot\cos[(\alpha/3)+120^\circ]+I_1/3 \tag{3.1.2.7}$$
$$\sigma_3=2S\cdot\cos[(\alpha/3)+240^\circ]+I_1/3$$

with $S=(R/3)^{1/2}$ and $\alpha=\cos^{-1}(-Q/2T)$

and $R=\frac{1}{3}I_1^2-I_2$, $\quad Q=\frac{1}{3}I_1\cdot I_2-I_3-\frac{2}{27}I_1^3$, $\quad T=\left(\frac{1}{27}R^3\right)^{1/2}$

After calculation of the three principal stress values, they can be placed in their normal conventional order of magnitude.

Evaluation of direction cosines for principal stresses (eigen vectors)

Having determined the three principal stress values for a given three-dimensional complex stress state using the above procedure, a complete solution of the problem generally requires that we determine the directions in which these stresses act – as given by their respective direction cosines or given eigen vector values.

The relationship between a particular principal stress σ_p (p = 1, 2, 3) and the Cartesian stress components is given by Eqs. (3.1.5b):

$$\begin{aligned} 0 &= (\sigma_x - \sigma_p)\cdot\cos\alpha + \tau_z\cdot\cos\beta + \tau_y\cdot\cos\gamma \\ 0 &= \tau_z\cdot\cos\alpha + (\sigma_y - \sigma_p)\cdot\cos\beta + \tau_x\cdot\cos\gamma \\ 0 &= \tau_y\cdot\cos\alpha + \tau_x\cdot\cos\beta + (\sigma_z - \sigma_p)\cdot\cos\gamma \end{aligned} \qquad (3.1.2.8a,b,c)$$

If one of the known principal stress values, say σ_1, is substituted in the above equations together with given Cartesian stress components, the three equations result in the three unknown direction cosines for that principal stress, i.e., $\cos\alpha_1$; $\cos\beta_1$ and $\cos\gamma_1$. However, only two of these are independent equations, and the additional identity:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad (3.1.2.8d)$$

is required in order to evaluate $\cos\alpha_1$; $\cos\beta_1$ and $\cos\gamma_1$.

This procedure can be repeated, substituting the other principal stress values σ_2 and σ_3, in turn, to produce eigen vectors of these stresses.

3.2 Geometrical theory of strains

3.2.1 Displacement and strain components, and the relation between them

Let us take an elastic body and fix it in such a way as to prevent its displacement as an absolute rigid body. Then displacement of each of its points will be caused only by deformation, Fig.3.2.1.1.

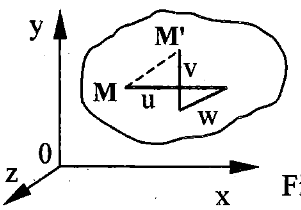


Fig.3.2.1.1

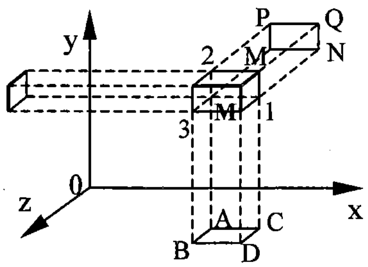


Fig.3.2.1.2

Consider any point *M(x,y,z)* in the body fixed as indicated above. The point *M* will be displaced into a new position *M'* as a result of the deformation produced. We designate the projections of the displacements *MM'* on the coordinate axes by ***u, v, w*** ; and since the displacements vary from point to point, the *projections of the displacements* are *functions of position*

$$u = f_1(x, y, z);\quad v = f_2(x, y, z);\quad w = f_3(x, y, z).$$

Let us now pass from **displacements to deformations**. In an elastic body, we isolate an infinitesimal with edges *dx, dy, dz*. During the deformation of the body it will displace and deform: the length of its edges will change and the initially right angles between the faces will distort.

To estimate the deformation of the elastic body at a given point *M* it is necessary to examine the elongations (linear deformations) of the edges *dx, dy, dz* of the isolated parallelepiped and the distortions of the angles *1M2, 1M3, 2M3* (shears or angular deformations). For this purpose consider the projections of the parallelepiped in the coordinate planes (Fig.3.2.1.2); obviously, the deformation of the parallelepiped itself can be deduced from the deformation of these three projections. In future we shall restrict ourselves to very small deformations, in the case of which the subsequent treatment may be greatly simplified.

Take, for instance, the projection of the element *M123* on the plane *Oxy*, Fig.3.2.1.2, where it will create a rectangular *MNPQ*. Before deformation, the lengths of the edges are: *MN = dx*, *MP= dy*. After deformation, they occupy the positions *M'N'* and *M'P'*, respectively. We now fix our attention on the projection of *MN*. If the displacement of the point *M* along the axis *Ox* is ***u***, the corresponding displacement if the point *N* is $u+\delta u = u + \frac{\partial u}{\partial x}dx$, where δu is the increment; since point *N* differs from point *M* only by the coordinate *x*, the small increment δu in the last formula is replaced, to infinitesimal quantities of the second order, by the partial derivation of function ***u*** with respect to the variable *x*.

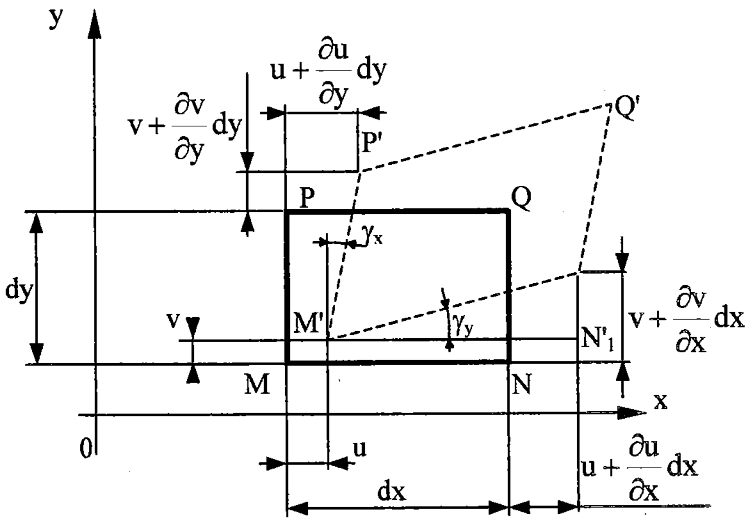


Fig.3.2.1.3

Likewise, if the displacement of point *M* along the axis *Oy* is ***v***, the displacement of point *N* along the same axis is expressed as $v+\frac{\partial v}{\partial x}dx$. The projection of the absolute elongation of the segment *MN* on the axis *Ox* being $\delta(dx) = \frac{\partial u}{\partial x}dx$, we obtain

$$\varepsilon_{xx} = \varepsilon_x = \frac{\delta(dx)}{dx} = \frac{\partial u}{\partial x}; \quad \varepsilon_{yy} = \varepsilon_y = \frac{\partial v}{\partial y}; \quad \varepsilon_{zz} = \varepsilon_z = \frac{\partial w}{\partial z} \qquad (3.2.1.1a,b,c)$$

for the *unit elongation (**normal strain**, extensional strain, extension)* of this edge (and by reasoning analogously, for the unit elongation of edge *MP* directed along axis *Oy* and in the same way, for the edge parallel to axis *Oz*, i.e., edge *M3,* in Fig.3.3.1.2). Thus, we have obtained the *formulas of **linear** deformations* (elongations) at a given point *M* of the body in the direction of the three coordinate axes.

Let us now turn to the analysis of ***angular*** *deformations*. We easily find the angle rotation $\alpha_{yx} \approx \tan\alpha_{yx} = \frac{B'B''}{A'B''} = \frac{\frac{\partial v}{\partial x}dx}{dx+\frac{\partial u}{\partial x}dx} = \frac{\frac{\partial v}{\partial x}}{1+\frac{\partial u}{\partial x}}$. Since we have confined ourselves to the case of very small deformations, we may omit the quantity $\varepsilon_{xx} = \frac{\partial u}{\partial x}$ in the denominator of the last expression as negligibly small compared with unity; we get $\gamma_{yx} = \frac{\partial v}{\partial x}$.

Similarly, we obtain the angle of rotation of the edge *MP* = *dy* in the plane *Oxy:* $\gamma_{xy} \approx \tan\gamma_{xy} = \frac{\partial u}{\partial y}$.

We now can easily find the shearing strain, i.e., the distortion of the right angle *NMP*:

$$\gamma_z = \gamma_{yx} + \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad (3.2.1.2a)$$

Note: Formula (3.2.1.2*a*) gives the expression of the shear angle occurring in the plane *Oxy* which is perpendicular to the *Oz* axis (thus resulting in the γ_z denotation).

Similarly, we obtain the expressions of shearing strains in the other two coordinate planes by a cyclic change between the letters which denote the displacements with respect to the applied coordinate planes.

Collecting together the above results, we get six basic relations characterising deformation:
a/ unit elongations (extensional strains or extensions)

$$\varepsilon_x = \frac{\partial u}{\partial x}; \quad \varepsilon_y = \frac{\partial v}{\partial y}; \quad \varepsilon_z = \frac{\partial w}{\partial z} \qquad (3.2.1.3a,b,c)$$

b/ shearing strains

$$\begin{aligned} \gamma_x &= \gamma_{zy} + \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}; \\ \gamma_y &= \gamma_{xz} + \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}; \\ \gamma_z &= \gamma_{yx} + \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}. \end{aligned} \qquad (3.2.1.4a,b,c)$$

3.2.2 Tensor character of the strain at a given point in a body

In the last section we introduced the concept of the stress tensor: from the viewpoint of the general theory of tensors, the formal and at the same time basic feature of the tensor character of a state of stress at a given point is that, as one passes from the coordinate planes (*Oxy, Oyz, Ozx*) to an arbitrary inclined plane ρ with outward normal ***n*** (having angles α, β, γ with the respective coordinates *x, y, z*), the stress components υ_x , υ_y , υ_z of the total stress υ, which is exerted on plane ρ, are expressed by Eqs.(3.1.1.1*a,b,c*). These formulas are linear with respect to the original components as well as to the direction cosines. Under a complete transformation involving the passage from axes *x, y, z* to new axes *x', y', z'* (one axis - say *y'*- of which coincides with the normal *n* of the above mentioned arbitrary inclined plane ρ, i.e., *y'* ≡ *n*) the components of the state of stress are expressed in terms of the original components $\sigma_x, \sigma_y, \sigma_z, \tau_x, \tau_y, \tau_z$ by projecting components υ_x , υ_y , υ_z of the total strain υ, which is exerted in the plane ρ, on the *y'* ≡ *n* axis according to Eq.(3.1.1.2). This expression is linear with respect to the original components and quadratic (or so-called bilinear) with respect to the direction cosines of the new system. The other two stress components exerted in this arbitrary inclined plane ρ are shearing stresses $\tau_{x'}$, $\tau_{z'}$ that can be obtained:

a/ <u>either</u> by projecting the components υ_x, υ_y, υ_z of the total strain υ, which is exerted in the plane ρ, on the x' and z' axes, respectively (these axes must lie in the plane ρ and form a rectangular coordinate system with the $y' \equiv n$ axis);

b/ <u>or</u> by projecting the total τ_ρ exerted in plane ρ on the x' and z' axes. The shearing strain τ_ρ is obtained by applying Pythagoras' theorem, Eq.(3.1.1.3).

In this section we shall show that the deformation of a body at a given point, determined by *nine* components of the strain matrix, Eq.(3.2.2.1), is also a tensor from the above point of view. The deformation at a given point in a body will be completely determined if we calculate the unit deformation of any infinitesimal segment drawn from the given point. Therefore, consider such a element $MR = dr$, Fig. 3.2.2.1.

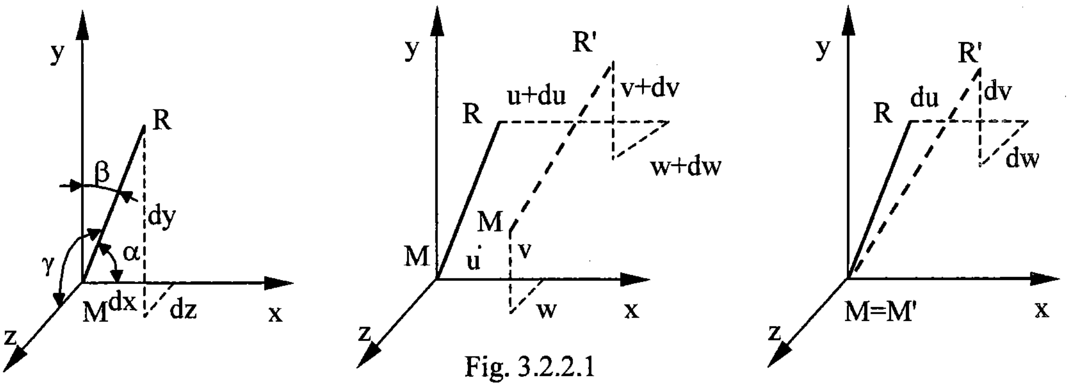


Fig. 3.2.2.1

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{vmatrix} = \begin{vmatrix} \varepsilon_x & \frac{\gamma_z}{2} & \frac{\gamma_y}{2} \\ \frac{\gamma_z}{2} & \varepsilon_y & \frac{\gamma_x}{2} \\ \frac{\gamma_y}{2} & \frac{\gamma_x}{2} & \varepsilon_z \end{vmatrix} \qquad (3.2.2.1)$$

The direction angles related to the coordinate axes x, y, z of this initial segment MR (before loading) are α, β, γ, respectively. After loading, the point M is shifted into position M', and R into R'. We denote $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ the displacement components of M, and $\boldsymbol{u + du}$, $\boldsymbol{v + dv}$, $\boldsymbol{w + dw}$ those of R. To be able to examine the elongation of segment MR more easily, we shift its deformed shape $M'R'$ in such a way that the points M and M' coincide. When expressing the total elongation $\boldsymbol{\Delta dr}$ of segment MR, it follows (when projecting the displacement increments $\boldsymbol{du}$, $\boldsymbol{dv}$, $\boldsymbol{dw}$ of the point R in the direction of the segment MR): $\Delta dr = du \cdot \cos\alpha + dv \cdot \cos\beta + dw \cdot \cos\gamma$ since we can neglect the changes of the direction angles α, β, γ, as very small quantities of higher order.

The unit elongation is then expressed as

$$\varepsilon = \frac{\Delta dr}{dr} = \frac{du}{dr} \cdot \cos\alpha + \frac{dv}{dr} \cdot \cos\beta + \frac{dw}{dr} \cdot \cos\gamma \quad . \qquad (3.2.2.2)$$

The displacement infinitesimal increments $\boldsymbol{du}$, $\boldsymbol{dv}$, $\boldsymbol{dw}$ may be replaced by their differentials (omitting small quantities of higher order):

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz;\quad dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz;\quad dw=\frac{\partial w}{\partial x}dx+\frac{\partial w}{\partial y}dy+\frac{\partial w}{\partial z}dz \qquad (3.2.2.3a,b,c)$$

It can readily be found from Fig.3.2.2.1 that it holds:

$$\cos\alpha=\frac{dx}{dr};\quad \cos\beta=\frac{dy}{dr};\quad \cos\gamma=\frac{dz}{dr}\quad . \qquad (3.2.2.4a,b,c)$$

After substituting these expressions into Eq.3.2.2.2, when considering Eqs.3.2.2.3*a,b,c* and 3.2.2.4*a,b,c*, we obtain

$$\varepsilon=\varepsilon_x\cdot\cos^2\alpha+\varepsilon_y\cdot\cos^2\beta+\varepsilon_z\cdot\cos^2\gamma+\gamma_x\cdot\cos\beta\cdot\cos\gamma+\gamma_y\cdot\cos\alpha\cdot\cos\gamma+\gamma_z\cdot\cos\alpha\cdot\cos\beta \qquad (3.2.2.5)$$

When comparing Eqs.3.1.1.3 and 3.2.2.5, we learn that these expressions are quite analogous. The quantities σ correspond with ε and τ with $\gamma/\mathbf{2}$. It can be proved that such an analogy holds generally. Therefore, all the conclusions issuing from the analysis of **stress** also hold for **strain.**

For instance, the magnitudes of the *principal unit elongations* are the roots of the following cubic equation:

$$\varepsilon^3-(\varepsilon_x+\varepsilon_y+\varepsilon_z)\cdot\varepsilon^2+\left(\begin{vmatrix}\varepsilon_y & \frac{\gamma_x}{2}\\ \frac{\gamma_x}{2} & \varepsilon_z\end{vmatrix}+\begin{vmatrix}\varepsilon_z & \frac{\gamma_y}{2}\\ \frac{\gamma_y}{2} & \varepsilon_x\end{vmatrix}+\begin{vmatrix}\varepsilon_x & \frac{\gamma_z}{2}\\ \frac{\gamma_z}{2} & \varepsilon_y\end{vmatrix}\right)\cdot\varepsilon-\begin{vmatrix}\varepsilon_x & \frac{\gamma_z}{2} & \frac{\gamma_y}{2}\\ \frac{\gamma_z}{2} & \varepsilon_y & \frac{\gamma_x}{2}\\ \frac{\gamma_y}{2} & \frac{\gamma_x}{2} & \varepsilon_z\end{vmatrix}=0\ \Rightarrow\ \varepsilon^3-I_1\cdot\varepsilon^2+I_2\cdot\varepsilon-I_3=0 \qquad (3.2.2.6)$$

3.3 Differential equations of equilibrium

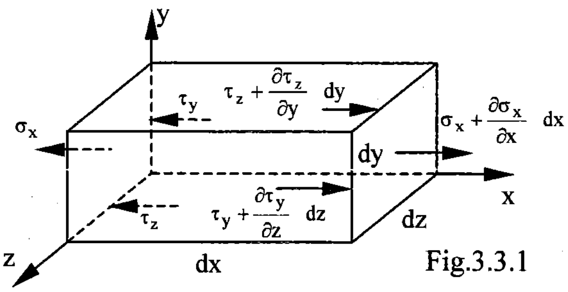


Fig.3.3.1

Imagine that we isolate in a solid an infinitesimal element (parallelepiped), the edges of which are ***dx, dy, dz***. (The stresses acting upon its faces generally vary according to their positions in the coordinate system. In the equilibrium equation of the forces acting upon the element in the *x*-coordinate direction is:

$$\left(\sigma_x+\frac{\partial\sigma_x}{\partial x}dx\right)\cdot dy\cdot dz-\sigma_x\cdot dy\cdot dz+\left(\tau_z+\frac{\partial\tau_z}{\partial y}dy\right)\cdot dx\cdot dz-\tau_z\cdot dx\cdot dz+ \\ +\left(\tau_y+\frac{\partial\tau_y}{\partial z}dz\right)\cdot dx\cdot dy-\tau_y\cdot dx\cdot dy+X\cdot dx\cdot dy\cdot dz=0 \qquad (3.3.1)$$

where X is the x-coordinate component of so called body forces applied to unit mass of the body; such are, for instance, the gravity, or centrifugal, or magnetic, forces and forces of inertia.

Eq.3.3.1 can be rearranged thus having

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_z}{\partial y}+\frac{\partial\tau_y}{\partial z}+X=0 \qquad (3.3.2a)$$

Analogously, based on equilibrium equations in the *y/z*-coordinate directions, we have

$$\frac{\partial\tau_z}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_x}{\partial z}+Y=0\quad /\quad \frac{\partial\tau_y}{\partial x}+\frac{\partial\tau_x}{\partial y}+\frac{\partial\sigma_z}{\partial z}+Z=0 \qquad (3.3.2b,c)$$

4. Torsion of bars with non-circular profiles

When twisting a shaft (with a circular profile), its originally plane cross-sections remain plane after loading (after shaft deformation). This is because the *shearing stresses* τ (produced by torsion), acting along all the so-called *shearing lines* (which are circular in shape), are *parallel.* (We know that τ is a tangent to these shearing lines). It can be seen (Fig.4.1) that the shearing stresses τ, acting along each shearing line, are in proportion. It follows from this that the corresponding *shearing strains* γ must also be proportional and, therefore, the profile does not deplane.

However, this does not hold when twisting a bar that does not have a circular profile. *As a consequence of this, the originally plane cross-sections* ***warp (deplane)*** (Fig. 4.2 and Fig. 4.3).

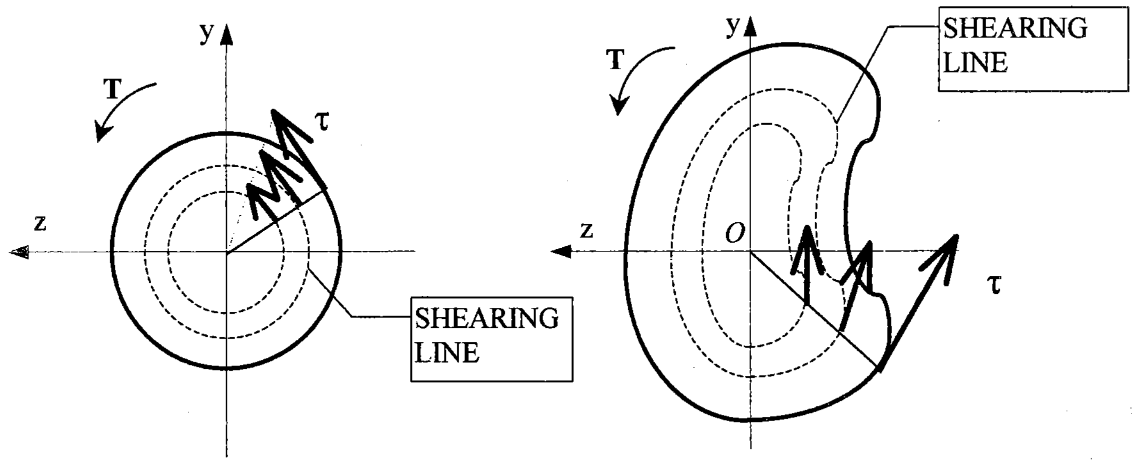


Fig.4.1 Fig.4.2

Based on the concept of warping (deplaning) there are two types of torsion:

When all the cross-sections of a twisted bar can warp (deplane) freely, we refer to ***"free (simple) torsion"***; while if free warping is restrained, e.g., by a fixation or by ribs, we refer to ***"restrained torsion"*** or ***"bending torsion"***.

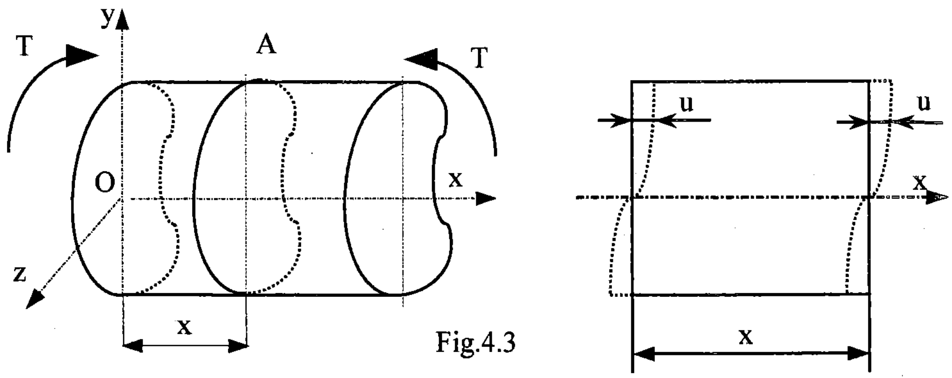


Fig.4.3

Denoting the relative twisting angle (the relative angular displacement of two cross-sections that are one unit apart) as ϑ, then the angular displacement of two arbitrary sections separated by the distance x (Fig.4.3) is $\varphi = \vartheta \cdot x$. Based on Fig.4.4, displacements v and w can be expressed:

$$v = -\overline{MM'} \cdot \cos\alpha = -r \cdot \varphi \cdot \cos\alpha = -r \cdot \vartheta \cdot x \cdot \cos\alpha = -\vartheta \cdot x \cdot z \qquad (4.1a)$$

$$w = \overline{MM'} \cdot \sin\alpha = r \cdot \varphi \cdot \sin\alpha = r \cdot \vartheta \cdot x \cdot \sin\alpha = \vartheta \cdot x \cdot y \qquad (4.1b)$$

(where the geometric relations: $\sin\alpha = \frac{y}{r}; \cos\alpha = \frac{z}{r}$, are readily obtained from Fig.4.4.)

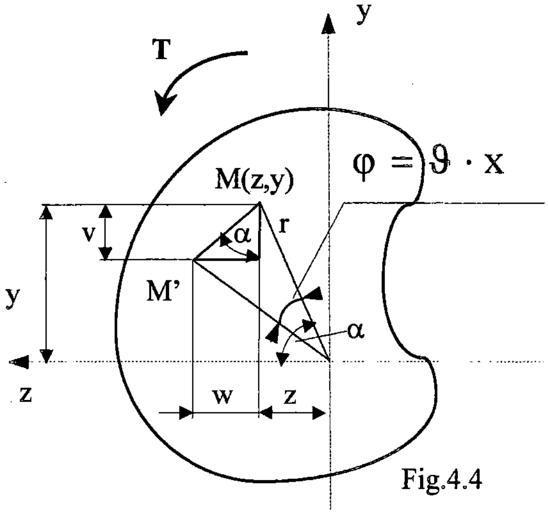


Fig.4.4

Applying to Eqs.(4.1*a,b*) the relations (3.2.1.3*a,b,c*) and (3.2.1.4*a,b,c*), Chap.3, we have successively:

$$\varepsilon_x = \frac{\partial u}{\partial x} = 0$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = 0$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0 \qquad (4.2a,b,c)$$

Note: Eq.(4.1.2*a*), i.e., $\varepsilon_x = 0$, follows from the fact that the deplanation, i.e., axial displacement, u = f(y,z), does not depend on its axial position.

$$\gamma_x = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\vartheta x + \vartheta x = 0 \ ; \ \gamma_y = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \vartheta y + \frac{\partial u}{\partial z} \neq 0 \ ; \ \gamma_z = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} - \vartheta z \neq 0 \qquad (4.3a,b,c)$$

Eqs.(4.3 *b,c*) represent *relations between strains (γ) and displacements (u).* Confronting them with the *circular shaft torsion* derivation (studied last term, cf. [1], Chap.8), we can denote all the expressions derived so far, as **step 1.** Following the analogy of the shaft torsion derivation we approach **step 2**, i.e., *the generalised Hooke's law* (GHL) application:

Substituting Eqs. (4.2*a,b,c,*) and (4.3*a,b,c*) into GHL (cf. [1], Chap.5, Eqs.5.8.2.1*a,b,c* and 5.8.2.2*a,b,c*), it follows that

$$\sigma_x = \sigma_y = \sigma_z = \tau_x = 0 \quad (a,b,c,d) \quad \text{and} \quad \tau_y = G(\vartheta y + \frac{\partial u}{\partial z}) \neq 0 \quad (e) \qquad \tau_z = G(\frac{\partial u}{\partial y} - \vartheta z) \neq 0 \quad (f) \qquad (4.4a,b,c,d,e,f)$$

These results show that, when twisting a non-circular profile bar, no normal stresses arise and the shearing stress has a general direction in the cross-sections (it follows from the theorem *of associated shearing stresses* that the shearing stress components τ_y and τ_z have their counterparts in the bar axial sections).

Differentiating Eqs. (4.4*e,f*) with respect to the corresponding coordinates, and then subtracting the first from the second, we obtain successively a partial differential equation:

$$\frac{\partial \tau_y}{\partial y} = G(\vartheta + \frac{\partial^2 u}{\partial y \partial z}), \quad \frac{\partial \tau_z}{\partial z} = G(\frac{\partial^2 u}{\partial y \partial z} - \vartheta) \quad \Rightarrow \quad \frac{\partial \tau_z}{\partial z} - \frac{\partial \tau_y}{\partial y} = -2G\vartheta \qquad (4.5)$$

as a result of combining the 1st and 2nd steps.

Step 3: *Equilibrium equations*

In Chapter 3 (3D-stress state), we derived the partial differential Eqs.(3.3.2*a,b,c*) of a body element equilibrium. For non-circular profile torsion, only the first of these equations will be used:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_z}{\partial y}+\frac{\partial \tau_y}{\partial z}+X=0$$

When twisting a rod, the volume force $X = 0$ and, based on the fact that $\sigma_x = 0$, also $\frac{\partial \sigma_x}{\partial x}=0 \Rightarrow$

$$\frac{\partial \tau_z}{\partial y}+\frac{\partial \tau_y}{\partial z}=0 \qquad (4.6)$$

4.1 Stress function

In order to assess shearing stress, we have a system of two partial differential Eqs. (4.5) and (4.6). No general solution of this eq. system is possible (solutions - and rather complex ones - exist for several particular cases). Theory of elasticity teaches that, if a solution of a strength problem is found, it will be unique. We shall try to find a function $F = F(y,z)$, the partial derivatives of which (Eqs.4.1.1*a,b*) give the values of the shearing stresses

$$\tau_y = \pm\frac{\partial F}{\partial y} = -\frac{\partial F}{\partial y}\ ; \quad \tau_z = \mp\frac{\partial F}{\partial z} = +\frac{\partial F}{\partial z} \qquad (4.1.1a,b)$$

A function which possesses the mentioned characteristic is called the "stress function". Geometrically, we can consider $F(y,z)$ as a surface "inflated" over the bar cross-section, Fig.4.1.1. The above introduced partial derivatives denote the slope magnitudes of the tangents t_1, t_2, respectively:

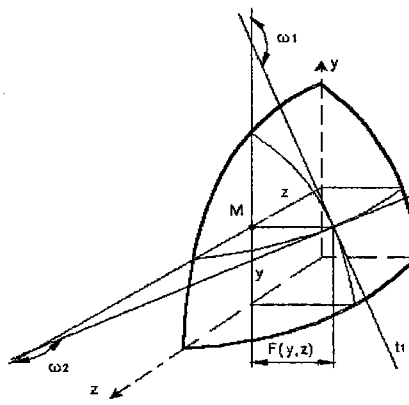


Fig.4.1.1

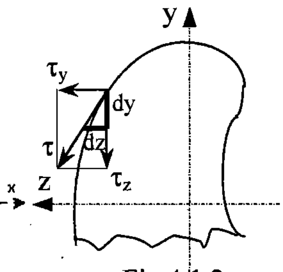


Fig.4.1.2

$$\mathrm{tg}\varpi_{1(2)} = \frac{\partial F}{\partial y(z)} < 0$$

(which are negative for the concave surface, where $\varpi_{1,2} = \pi/2$).

Note: It is evident from Fig.4.1.2 that $\tau_y > 0$ (having a positive sense with respect to the z axis) and $\tau_z < 0$ (being directed oppositely with respect to the y axis). In order to respect these two facts we take the lower signs in Eqs.(4.1.1*a,b*).

After respective differentiating, Eqs.(4.1.1*a,b*) yield

$$a)\ \frac{\partial \tau_y}{\partial y} = -\frac{\partial^2 F}{\partial y^2}\ ; \quad b)\ \frac{\partial \tau_y}{\partial z} = -\frac{\partial^2 F}{\partial y \partial z}; \quad c)\ \frac{\partial \tau_z}{\partial y} = +\frac{\partial^2 F}{\partial y \partial z}; \quad d)\ \frac{\partial \tau_z}{\partial z} = +\frac{\partial^2 F}{\partial z^2}$$

Two of these expressions (*b* and *c*) satisfy Eq.(4.6), while applying the remaining expressions (*a* and *d*) in Eq.(4.5) we obtain

$$\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}=-2\vartheta G\text{; (since } F=F(y,z)\Rightarrow \frac{\partial^2 F}{\partial x^2}=0\text{) we can write: } \nabla_1^{\,2}F=-2G\vartheta \qquad (4.1.2)$$

(where we applied the operator $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$)

As a consequence of the **complementary shearing stress theorem,** <u>the shearing stress direction has to be tangent to the cross-section contour,</u> Fig.4.1.2. Considering there the similar triangles (formed by shearing stress components τ_y, τ_z and by differentials *dy, dz*), we have $\frac{dy}{dz}=\frac{\tau_z}{\tau_y}$, where we substitute from Eq.(4.1.1*a,b*) and thus obtain the boundary condition of the stress function:

$$\frac{\tau_z}{\tau_y}=\frac{dy}{dz}=\frac{\frac{\partial F}{\partial z}}{-\frac{\partial F}{\partial y}},\qquad \text{or}\qquad \frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial z}dz=0 \qquad (4.1.3)$$

Note 1: Where the left hand side is a *total differential*, Eq.(4.1.3) can be rewritten as $dF = 0$, which (after integration) yields F = const. Thus, <u>the **boundary condition** that the *stress function F* has a constant value along the cross-section contour</u>. If the cross-section contour is demarcated by several curves, i.e., a twisted rod has longitudinal holes, the *stress function F* must be constant along all the cross-section contours.

Note 2: Stress function assessment is generally not a simple problem,but when the profile contour equation (Fig.4.1.2) obeys the following condition $\nabla_1^2 f = C$, where *C* denotes a non-zero constant value, we can determine the stress function easily, being $F(y,z)=-\frac{2G\vartheta}{C}f(y,z)$, which can be proved when considering $\nabla_1^2 F=-\frac{2G\vartheta}{C}\nabla_1^2 f$ and, because $\nabla_1^2 f = C$, the basic condition Eq.(4.1.2) is fulfilled: $\nabla_1^2 F=-2G\vartheta$.

Exact stress functions can be obtained for the following profiles: i) *elliptic,* and, ii) *equilateral triangle.*

4.2 Characteristics of the stress function F(y,z)

1/ Function ***F(y,z)*** can also be referred to as a **stress-cap** (SC)**.** In Fig.4.2.1, the stress-cap is cut with a plane $x = h$. In the front view, the intersection line ($F(y,z) = h$), i.e., the *isohypse*, successively obtains the forms (cf. Eqs.4.1.1*a,b* and 4.1.3):

$$dF=\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial z}dz \quad\text{and}\quad \frac{dy}{dz}=-\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y}}=\frac{\tau_z}{\tau_y} \qquad (4.2.1)$$

Therefore *the **stress-cap isohypses** represent the **shearing lines** (including the contour line, where h = 0).*

2/ When we cut the stress-cap with two planes, ***x = h*** and ***x = h + dh,*** which are infinitesimally close to each other, then the intersections of these two planes with the stress-cap form two infinitesimally close isohypses. From Fig.4.2.2, it follows successively

$$\tau=|\tau_y|\cos\beta+|\tau_z|\sin\beta,\qquad \text{or}\qquad \tau=\left|\frac{\partial F}{\partial y}\right|\cos\beta+\left|\frac{\partial F}{\partial z}\right|\sin\beta. \qquad (4.2.2)$$

Denoting the distance of the shearing lines as ***dn***,

it also holds that $\cos\beta = -\frac{dy}{dn}$, $\sin\beta = -\frac{dz}{dn}$, so that $\tau = -\left(\left|\frac{\partial F}{\partial y}\right|dy + \left|\frac{\partial F}{\partial z}\right|dz\right)\cdot\frac{1}{dn}$. (4.2.3)

Applying Eq.((4.3.1), we obtain $\tau = -\frac{dF}{dn}$ (4.2.4)

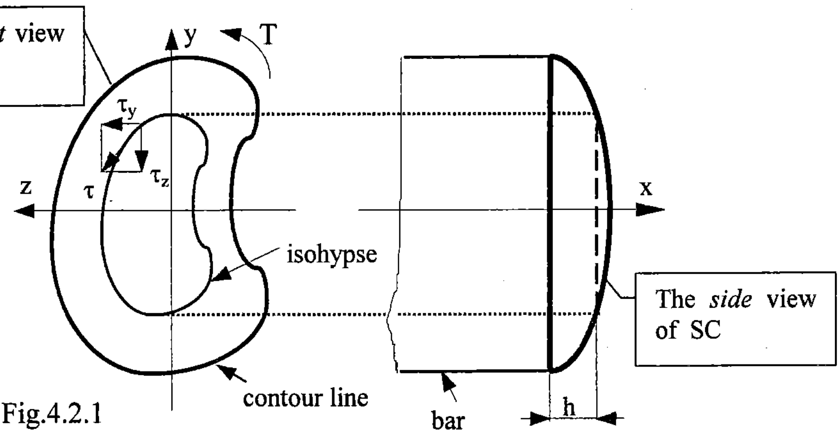


Fig.4.2.1

Note: The last expression implies two interpretations:

a/ The τ magnitude at a profile point is given by the stress-cup gradient at the respective point.

b/ When rearranging Eq.(4.2.4) we obtain

$$|\tau\cdot dn| = dF = \text{const} \quad (4.2.5)$$

which is called the ***continuity equation*** (since its shape is similar to the *continuity eq.* known from hydraulics). The *continuity equation* (Eq.4.2.5) expresses:

where the shearing lines (similar to the lines of flow) are concentrated, the shearing stresses (similar to the flow velocity) increase.

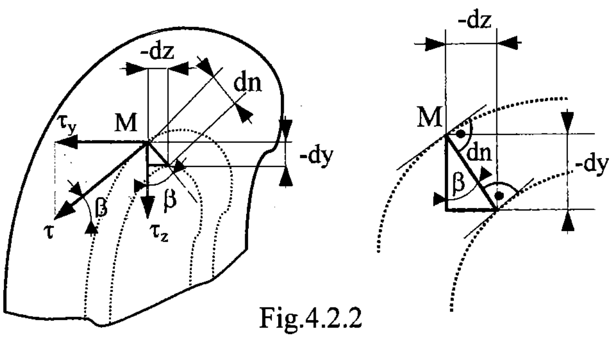


Fig.4.2.2

3/ Let us, as in the previous item, cut the stress-cap with two infinitesimally close planes and thus obtain two infinitesimally close *isohypses* (i.e., shearing lines), Fig.4.2.3. With respect to the origin of the coordinates *O* (i.e., the pole of the twist), an *elementary force*, i.e., $\tau dn \cdot ds$, produces an *elementary torque* of the second order: $d^2T = \tau dn \cdot ds \cdot \rho$,

so that the **elementary torque** (of the first order) carried by the belt between these two shearing lines under consideration (while integration is executed along the central line ***c***) is:

$dT = \int_c \tau dn \cdot 2dS$, (where $dS = \frac{1}{2}\rho ds$ is the area of the shaded triangle). Considering that it holds (according to the *second stress-function characteristic*): $|\tau dn| = dF = \text{const}$, the elementary torque can be expressed as follows: $dT = 2\tau dn \int_c dS$, or $dT = 2\tau dn S_c$. (4.2.6)

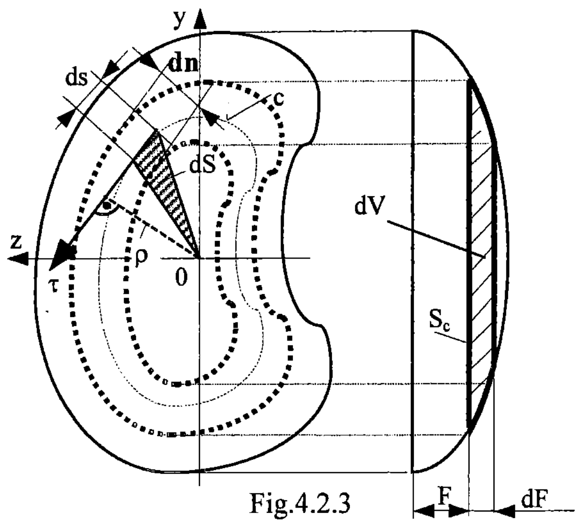


Fig.4.2.3

Substituting Eq.(4.2.5), i.e., $dT = 2S_c dF = 2dV$, we obtain by integrating: $T = 2V$, (4.2.7)
i.e., the *torque magnitude* borne by the given profile is equal to the two-multiple of the *stress-cap volume.*

Note: Eq.(4.2.6) is very important. From this point of solution we can continue in two modes:
I) continuing in the integration process and thus obtaining an allowed torque to be applied in torsion of a **solid profile**, see below;
II) interrupting the integration process when considering the similarity of the two infinitesimally close isohypses to a **hollow thin-walled profile** -while adopting Eq.(4.2.6) - (see later in section 4.3.1 where hollow thin-walled profiles will be discussed).

4.3 Thin-walled profiles

Based on the formulas obtained in the previous sections, problems of ***free torsion*** of ***thin-walled cross-sections*** can be derived. The thin-walled cross-sections are divided into two groups: i) ***hollow (closed),*** and, ii) ***open thin-walled profiles.*** We will learn that the *deplanation* of hollow profiles under *free torsion* is much lower (due to their high stiffness) than deplanation with open profiles. And, since in practice predominantly ***restrained torsion*** (restraining deplanation) takes place, only the results obtained here for free torsion of hollow profiles are of direct practical use, while the torsion formulas for open profiles need to be completed by the ***restrained torsion theory.***

4.3.1 Hollow thin-walled profiles

The shape of hollow thin-walled profiles (with wall thickness ***n***) being similar to the configuration given by the two thick-dotted isohypses (Fig.4.2.3) which are separated by a distance ***dn,*** we start from Eq.4.2.6 and, having transformed it (by an engineering approach) from elementary quantities to final quantities, we have

$$T = 2 \cdot \tau_c \cdot n_{min} \cdot S_c \qquad (4.3.1.1)$$

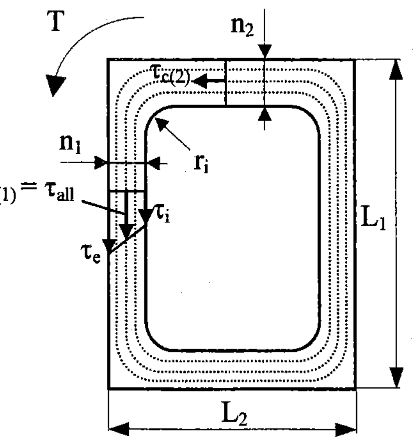


Fig.4.3.1.1

where $|\tau_c \cdot n_{min}| = \text{const}$, according to Eq.(4.2.5), and $\tau_c = \tau_{all}$ holds for central curve *C* in the thinnest part (n_{min}) of the profile. It is evident from Fig. 4.3.1.1 that $\tau_e > \tau_{all}$, which is inadmissible, requires a reduction of torque *T*. For this purpose we apply *Stoke's theorems.*

I) Stress reduction for a ***straight*** profile part:

1) Based on the supposed linear stress distribution (Fig.4.3.1.1): $\tau_e + \tau_i = 2\tau_{all}$

2) Elementary *Stoke's theorem*: (for a straight part $\rho \to 0$)

$$\frac{d\tau}{dn} + \frac{\tau}{\rho} = 2G\vartheta \Rightarrow \frac{\tau_e - \tau_i}{n_{min}} = 2G\vartheta \Rightarrow \tau_e - \tau_i = 2G\vartheta n_{min}$$

1) + 2) yields: $\tau_e = \tau_{all} + G\vartheta n_{min}$ (4.3.1.2)

3) General *Stoke's theorem*: $G\vartheta = \dfrac{\oint \tau ds}{2 \cdot S_c} \Rightarrow G\vartheta = \dfrac{\tau_{all} \cdot (L_1 - n_2) + \tau_{all} \frac{n_1}{n_2} \cdot (L_2 - n_1)}{(L_1 - n_2) \cdot (L_2 - n_1)}$

Note: From the ***continuity equation*** (the 2nd stress-cap characteristic), it follows $\tau_{all} \cdot n_1 = \tau_2 \cdot n_2$

II) Stress reduction at a sharp corner of the profile, Fig. 4.3.1.2: ($dn = d\rho$)

Elementary *Stoke's theorem*: $\dfrac{d\tau}{dn} + \dfrac{\tau}{\rho} = 2G\vartheta \Rightarrow$

$$\frac{d\tau}{d\rho} + \frac{\tau}{\rho} = 2G\vartheta / \cdot \rho \cdot d\rho \Rightarrow d(\tau \cdot \rho) = 2G\vartheta \cdot \rho \cdot d\rho$$

After integrating we have $\tau = G\vartheta\rho + \dfrac{C}{\rho}$. (4.3.1.3)

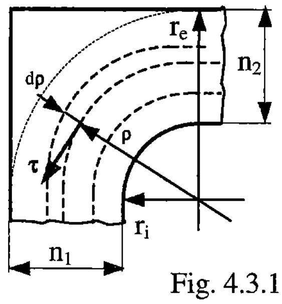


Fig. 4.3.1.2

Integration constant *C*, found by applying BC (the *continuity equation*):

$$\tau_{all} \cdot n_1 = \int_{r_i}^{r_e} \tau(\rho) \cdot d\rho, \text{ is } C = \frac{\tau_c n - \frac{G\vartheta}{2}\left(r_e^2 - r_i^2\right)}{\ln \frac{r_e}{r_i}}. \quad (4.3.1.4)$$

Stress concentration (with a sharp corner) arises at radius r_i: $\tau_{ri} = G\vartheta r_i + \dfrac{C}{r_i}$ (4.3.1.5)

The final value of allowable torque T_{all} that the profile can bear is obtained by reducing Eq.(4.3.1.1):

$T_{all} = T \cdot \dfrac{\tau_{all}}{\tau_{max}}$, where $\tau_{max} = \max(\tau_e, \tau_{ri})$, see Eqs.(4.3.1.2) and (4.3.1.3).

Note: After this reduction, the allowable stress τ_{all} will not be exceeded anywhere in the profile. **The formulas introduced here can be applied in practical design.**

4.3.2 Open thin-walled profiles

The theory of free torsion of open thin-walled profiles is based on formulas derived for a **long narrow rectangle**, Fig. 4.3.2.1:

In such a case, the stress-cap *F(y,z)* can be simplified into a parabolic cylinder *F(y)* (thick dotted line), and therefore Eq. (4.5) simplifies to $\dfrac{d\tau_y}{dy} = 2G\vartheta \Rightarrow \tau_y = 2G\vartheta y + C_1$ *).

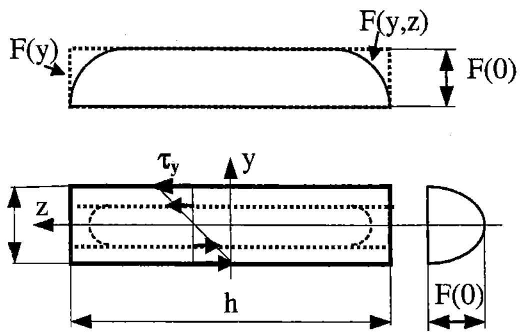


Fig. 4.3.2.1

From Eq.(4.1.1*a*): $\tau_y = -\dfrac{dF(y)}{dy}$, we have

$F(y) = -G\vartheta y^2 + C_2$.

Boundary conditions: 1/ $\tau_y(0) = 0 \Rightarrow C_1 = 0$;

2/ $F(b/2)=0 \Rightarrow$

$C_2 = G\vartheta (b/2)^2$;

yield *stress-function* $F(y) = G\vartheta \cdot \left(\frac{b^2}{4} - y^2\right)$ **),

twice the volume of which indicates the torque borne by the profile:

$T = 2V = 2 \cdot \frac{2}{3} \cdot F(0) \cdot h \cdot b = G\vartheta \cdot \frac{1}{3} b^3 h$ ***). From the *) expressions, we can express

$2G\vartheta = \frac{\tau_y(\frac{b}{2})}{\frac{b}{2}} = \frac{\tau_{max}}{b/2}$. In combination with ***) we have successively:

i) Strength criterion:; $\tau_{max} = \frac{T}{Z_T} = \frac{T}{\frac{1}{3} b^2 h} \le \tau_{all}$; (4.3.2.1)

ii) flexibility (the rate of angle of twist): $\vartheta = \frac{T}{G \cdot I_T} = \frac{T}{G \cdot \frac{1}{3} b^3 h}$ (4.3.2.2)

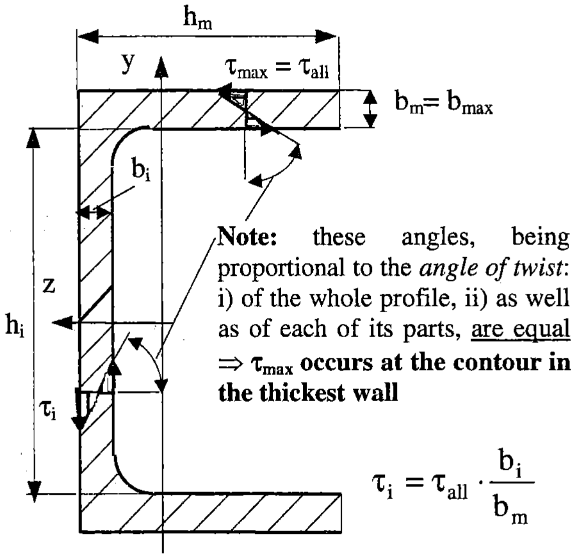


Fig.4.3.2.2

The resulting torque of a **channel** profile, Fig.4.3.2.2, is the sum of the torques the component rectangles can bear:

$$T = \sum_n T_i = \sum_n \tfrac{1}{3} b_i^2 h \cdot \tau_i =$$

$$= \sum_n \tfrac{1}{3} b_i^2 h \cdot \tau_{all} \cdot \frac{bi}{b_m}$$

$$T = Z_T \cdot \tau_{all} = \frac{I_T}{b_{max}} \cdot \tau_{all} \qquad (4.3.2.3)$$

where $I_T = \sum_n I_i = \sum_n \frac{1}{3} b_i^3 \cdot h_i$

<u>It is necessary to check for a *possible stress concentration* in ***sharp corners***</u> by applying Eq.(4.3.1.3), where integration constant $C = \frac{-\frac{G\vartheta}{2}\left(r_e^{\,2} - r_i^{\,2}\right)}{\ln \frac{r_e}{r_i}}$ obtains a slightly different shape by applying BC:

$0 = \int_{r_i}^{r_e} \tau(\rho) \cdot d\rho$ (resulting stress "flow" across the profile wall thickness is equal zero). The angle of twist to be substituted in Eq.(4.3.1.3), is obtained from expression **), where the torque calculated from Eqs.(4.3.2.3) is substituted. If $\tau_{ri} > \tau_{all}$ a reduction, $T_{all} = T \cdot \frac{\tau_{all}}{\tau_{max}}$, is needed.

Note: <u>*General Stokes' Theorem* applied for torsion</u>

When twisted, the rod cross-sections are deplaned in all their points *M(y,z)* by a displacement *u(y,z)* (in the *x* direction). The function *u(y,z)* is a coherent and univocal function of *y,z*. As the displacement change must be coherent as well, the total differential *du* is also a coherent function of *y,z*. In a considered point (Fig.1), we cut out an arbitrary, closed, simply continuous curve *C*, to each relevant

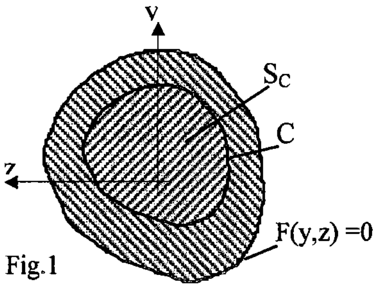


Fig.1

point of which is assigned a certain value *u(y,z)*, and thus also *du(y,z)*, which we integrate along the curve *C* $\oint_{(C)} du = \oint_{(C)} \left(\frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \right) = 0$.

It follows from Eqs.(4.1.4*e,f*): $\frac{\partial u}{\partial y} = \frac{\tau_z}{G} + \vartheta z$; $\frac{\partial u}{\partial z} = \frac{\tau_y}{G} - \vartheta y \Rightarrow$

$$\Rightarrow \oint_{(C)} (\tau_z dy + \tau_y dz) = G\vartheta \oint_{(C)} (ydz - zdy) \quad *)$$

Based on Fig.2 we have $\tau_z dy + \tau_y dz = \tau_t ds$. Since it holds, according to Fig.3, that $\oint_{(C)} ydz = S_C$ and $\oint_{(C)} zdy = -S_C$, resulting in $\oint_{(C)} (ydz - zdy) = 2S_C$, the right hand side integral in Eq.*) expresses twice the area covered inside curve *C*. Thus we rewrite Eq.*) into a simple shape $\oint_{(C)} \tau_t ds = 2G\vartheta S_C$ **)

which is called the *General Stokes Theorem for torsion.*

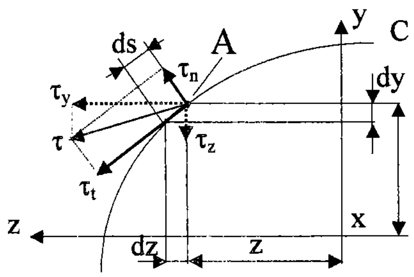


Fig.2

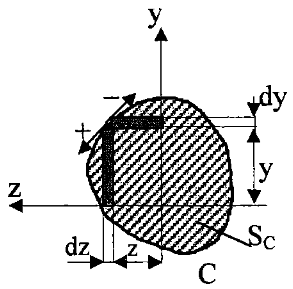


Fig.3

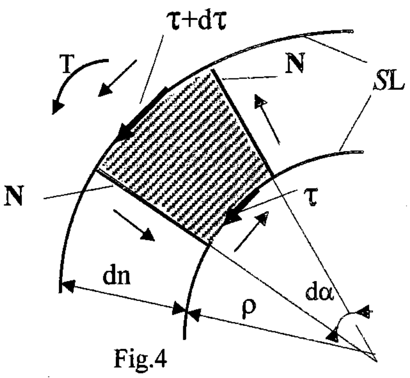


Fig.4

Elementary Stokes Theorem

Let us apply the *general Stokes Theorem* on an elementary area *dS* demarcated by two near shearing lines *SL* and two normals *N* containing an elementary angle *dα*, Fig.4. Then Eq.**) can be modified into $(\tau + d\tau) \cdot (\rho + dn) \cdot d\alpha - \tau \cdot \rho \cdot d\alpha = 2G\vartheta \cdot \rho \cdot d\alpha \cdot dn \Rightarrow \frac{d\tau}{dn} + \frac{\tau}{\rho} = 2G\vartheta$ which is called the *Elementary Stokes Theorem for torsion.*

It is used for determining the shearing stress distribution:

i) Over the wall thickness of hollow thin-walled profiles, where Eq.(7) will take an approximate shape

$$\frac{\tau_e - \tau_i}{n_{min}} = 2G\vartheta$$

ii) In the sharp corners of both hollow and open thin-walled profiles, where Eq.(7) obtains successively the shapes as follow

$$\frac{d\tau}{d\rho} + \frac{\tau}{\rho} = 2G\vartheta \,/\cdot \rho \cdot d\rho \Rightarrow \underbrace{d\tau \cdot \rho + \tau \cdot d\rho}_{d(\tau.\rho)} = 2G\vartheta \cdot \rho \cdot d\rho \Rightarrow \tau = G\vartheta \cdot \rho + \frac{C}{\rho},$$

where *integration constants:* $C = \frac{\tau_{all} n - \frac{G\vartheta}{2}\left(r_e^2 - r_i^2\right)}{\ln \frac{r_e}{r_i}}$, *or* $C = \frac{-\frac{G\vartheta}{2}\left(r_e^2 - r_i^2\right)}{\ln \frac{r_e}{r_i}}$,

(hollow, or open profile).

5. Buckling of columns

A long slender bar subjected to axial compression is called a *column*. The term *column* is frequently used **to describe a vertical member** (whereas the word *strut* is occasionally used in regard to inclined bars).

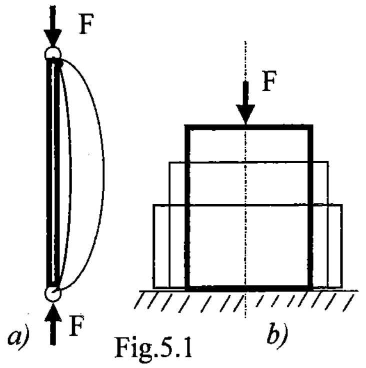


Fig.5.1

Type of failure of a column : Failure of a column occurs by *buckling*, i.e., by lateral deflection of the bar (Fig.5.1*a*). By contrast, it is to be noted that failure of a short compression member occurs by yielding of the material (Fig.5.1*b*). Buckling, and hence failure, of a column may occur even though the maximum stress in the bar is less than the yield point of the material. Though every carrying (load-bearing) system is deformed, in the problems studied so far, we were able to neglect the influence of small distortions of the systems on the state of their equilibrium. In such a way, the problems became linear, i.e., *elastic deformations were proportional to the external load.* For instance, we did not consider the influence of the *beam's* horizontal axis length shortage Δl, Fig.5.2*a*, when computing bending moments, shearing forces and stress distribution. Briefly, *we start from the basic geometry of the non-deformed system.* These simplifying assumptions cannot be applied when the loading force acts on the axis of a slender bar. A contingent deflection of the bar (caused for instance by a processed eccentricity) brings about supplementary bending moments $M = F{\cdot}y$, which additionally increase both stress and deflection (Fig.5.2*b*). *Deformations are already not proportional to the axial loading magnitude and their influence cannot be neglected.*

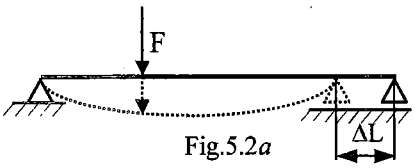


Fig.5.2*a*

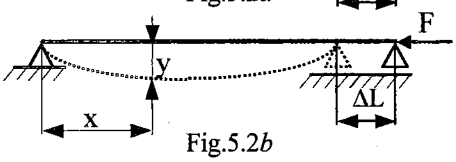


Fig.5.2*b*

Types of column stability

Consider an ideal case of a column loaded with an axial force *F* (*ideal*→ material homogeneity, column geometry, co-axial load, etc.), then three types of column stability can occur (Fig.5.3*a,b,c*):

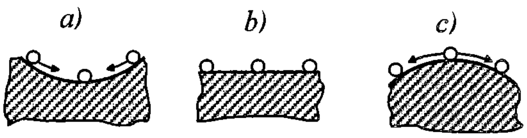


Fig.5.3

a) The force F is relatively small (later we shall find out that $F < F_{crit}$*).*

If the column is bent out by a short-term lateral force exertion, the column stiffness tends to settle the column to its original equilibrium position, which is reached after a short vibration of the column. The column will not be bent in the end. In this case the system is said to be *stable (in stable equilibrium).* In Fig. 5.3a, the concept of stable equilibrium is demonstrated by a sphere in a concave cup.

b) The force F is equal to the critical force F_{crit} $(F = F_{crit})$

If some lateral impulse bends the column out slightly, and the column remains in its new bent position *without any tendency to move out*, the system is said to be *in indifferent equilibrium.* The corresponding axial force *F* is then called *the critical force* F_{crit} .In Fig.5.3*b*, the concept of indifferent equilibrium is demonstrated by a sphere on a plane.

Note: **All deflection, although theoretically unbounded, must be considered very small,** otherwise the following mathematical description will not hold.

c) The force F is greater than the critical force F_{crit} *(*$F > F_{crit}$*)*

When the critical force is exceeded slightly, $F > F_{crit}$, and a lateral impulse bends the column out of a straight line, the column deflection will increase quickly until rupture of the column occurs at a limit force F_{lim}. The system is said to be *unstable.* (The system loaded with $F>F_{crit}$ could theoretically be *in unstable equilibrium* for some time, which can be demonstrated by a sphere on a convex cup, Fig.5.3*c*). The limit force F_{lim} does not differ much from the critical force F_{crit}, and therefore this is considered to be, in practice, the limit of the carrying capacity of the column.

Note: A complex of problems, when the carrying capacity of members is not determined by their *strength* but by their *stable state of equilibrium*, is known as *the stability of structures.* Stability problems also occur with thin-walled members, e.g., *plates* (both plane and curved), *shells* (pressure vessels subjected to external pressure), *beams* with *thin-walled profiles* (both open and closed) and beams that are *pressed, twisted, bent* or under a combination of these loadings, etc. Stability problems can concern both a whole structure and parts of a structure. Exact solutions of most stability problems are very difficult to obtain, and thus designers are often satisfied with only approximate solutions or with experimental findings (which are also very problematic to carry out).

5.1 Basic modes of buckling of columns

Buckling is one of the basic stability problems that designers very often have to face. The critical force magnitude F_{crit} depends on the column's: *dimensions, material (and its homogeneity) and the mode of the column end fixations*. With respect to the column end conditions, four basic modes are usually brought forward, (Fig.5.1.1*a,b,c,d*):

I) column with one free end *A* supporting a load *F* and one fixed (clamped) end *B*;

II) pin-ended column;

III) column with one fixed end *B* and one pin-connected end *A* supporting a load *F*;

IV) column with two fixed ends *A* and *B* supporting a load *F*.

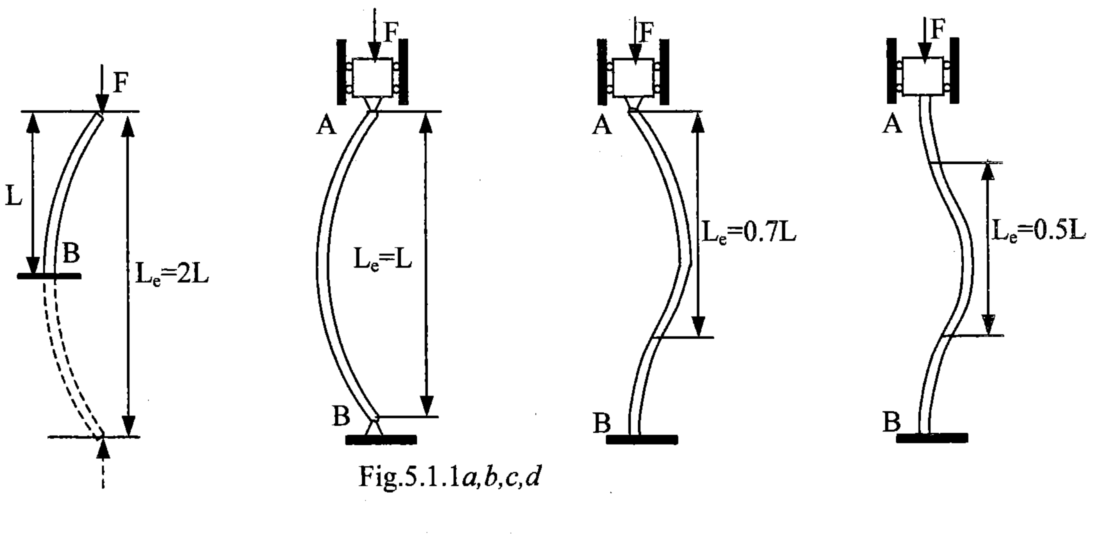


Fig.5.1.1*a,b,c,d*

5.1.1 Classic solutions

The elastic solution principle (in the range of validity of Hooke's law) for all the basic modes can be described as follows:

Consider the respective column critical loading *F (the symbol should be denoted accurately* F_{crit}) resulting in the column *indifferent equilibrium state*, i.e., an arbitrary lateral deflection $y_{(x)}$ of the column can take place, Fig.5.1.1.1 We are just to express two relevant mathematical relations:

1) Bending moment $M_{(x)}$, exerting in a given column point $\xi_{(x)}$ (at a distance x from end *B)* by applying the method of sections;

2) *Bernoulli's equation* $y'' = -\dfrac{M_{(x)}}{EI}$ applied to determine the beam deflection

Note: As the solution of the *column with the 2nd. type of end conditions* (briefly *2nd buckling mode*) will be proved to be the simplest, we will start with it.

2nd buckling mode (end conditions)

The relevant expressions for the 2nd *buckling mode* can readily be obtained from Fig.5

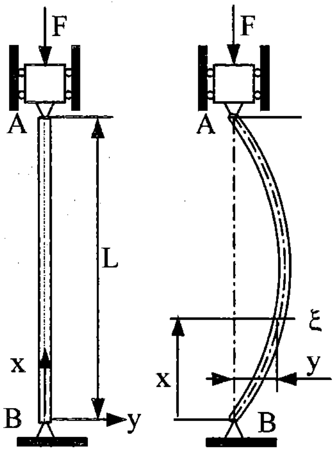


Fig.5.1.1.1

Ad 1) $M_{(x)} = F \cdot y$; ad 2) $y'' = -\dfrac{M_{(x)}}{EI} = -\dfrac{F}{EI} y$ introducing $\dfrac{F}{EI} = \alpha^2$ we obtain a homogeneous differential equation (of the second order, having constant coefficients): $y'' + \alpha^2 y = 0$, the solution of which is $y = A\cos\alpha x + B\sin\alpha x$ *end (boundary) conditions (BC):* for the *bottom pin* ...$x = 0; y_{(0)} = 0 = A\cos 0 + B\sin 0 \Rightarrow A = 0$ for the *upper pin* ... $x = L;\ y_{(L)} = 0 \Rightarrow B\sin\alpha L = 0$,which is satisfied:

i) either $B = 0$, then $y_{(x)} = 0$ ...and the column remains straight (*stable equilibrium*);

ii) or $\sin\alpha l = 0$, then $\alpha \cdot L = k \cdot \pi$, *) where $k = 1, 2, 3$...and the column will deflect: $y_{(x)} = B\sin\alpha x$ (***indifferent** equilibrium*).

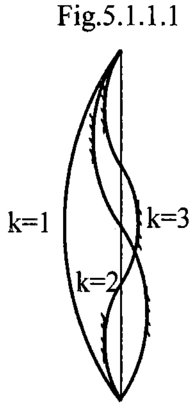


Fig.5.1.1.2

The minimum critical load will be delivered when substituting $k = 1$ into *).

Taking into account $\dfrac{F}{EI} = \alpha^2$, we obtain $\alpha \cdot L = L\sqrt{\dfrac{F}{EI}} = \pi$, and the critical load will be

$$F_{crit} = \frac{\pi^2 EI}{L^2} \qquad (5.1.1.1)$$

Note: The meaning of the different values of coefficient k is evident from Fig.5.1.1.2 (for application $k = 2;3...$, the column would need lateral supports).

Discussion of Eq.(5.1.1.1):

1. From formula (5.1.1.1) it is evident that the *lateral deflection* depends on the bending stiffness ***EI*** and for this reason the deflection *must set in the plane of the **lowest** bending stiffness* EI_{min}.
   [Therefore rods with sections (profiles) having considerable differences in *principal second moments of area* (cf. [1], Chap.9), e.g., channels (U-irons), are not advantageous for use as columns (or struts), because of their unbalanced buckling resistance, which means that they consume much material.]

2. The formula contains *Young's modulus of elasticity* ***E*** and therefore it has limited applicability. It can be applied only in the range of validity of *Hooke's Law*, i.e., only when the critical stress $\sigma_{crit} = F_{crit}/A$ does not exceed the proportional limit σ_{prop} of the applied material. The critical load is then rewritten:

$$F_{crit} = \frac{\pi^2 EI_{min}}{L^2} \qquad (5.1.1.1a)$$

The expression obtained is known as *Euler's formula*, after the Swiss mathematician Leonard Euler (1707-1783)

1st buckling mode (end conditions)

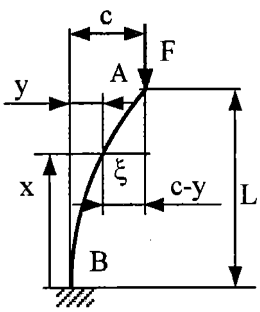


Fig.5.1.1.3

Based on Fig.5.1.1.2, it holds:

$$1)\, M_{(x)} = -F(c-y);\; 2)\, y'' = -\frac{M_{(x)}}{EI} = \frac{F}{EI}(c-y) \quad \Rightarrow$$

$$y'' + \alpha^2 y = \alpha^2 c$$

The particular integral being $y_p = c$, we obtain the solution in the following shape $y = A\cos\alpha x + B\sin\alpha x + c$;

$y' = -\alpha A\sin\alpha x + \alpha B\cos\alpha x$ (needed for BC)

End (boundary) conditions:

for bottom fixing... $x = 0 : y_{(0)} = 0 = A\cos 0 + B\sin 0 + c \quad \Rightarrow \quad A = -c$

and $y'_{(0)} = 0 = -\alpha \cdot A\sin 0 + \alpha \cdot B\cos 0 \quad \Rightarrow \quad B = 0$

Thus, the deflection curve equation is $y = -c \cdot \cos\alpha x + c = c(1-\cos\alpha x)$, which holds for the whole column and therefore also for the upper free end... $x = L : y_{(1)} = c = c(1-\cos\alpha L) \Rightarrow c \cdot \cos\alpha L = 0$

To satisfy the last condition:

i) either c = 0, then $y_{(x)} = 0$ ... the column remains straight (*stable* equilibrium);

ii) or $\cos\alpha l = 0$, then $\alpha \cdot L = k \cdot \frac{\pi}{2}$ *), where *k = 1, 2, 3...*, and the column will deflect (***indifferent equilibrium***), obeying the relation: $y = c(1-\cos\alpha x)$

The minimum critical load will be delivered when substituting *k = 1* (any other *k* value would require lateral supports), into *): $\alpha L = L\sqrt{\frac{F}{EI}} = \frac{\pi}{2}$ and the critical load will be $F_{crit} = \frac{\pi^2 EI_{min}}{4L^2}$ (5.1.1.2)

Note: Another approach to this 1st *buckling mode* solution can be:

We may observe that the column will behave as the upper part of a pin-connected column (Fig.5.1.1*b*). The critical load for the column in Fig.5.1.1*a* is thus the same as for the pin-ended column of Fig.5.1.1*b* and may be obtained from Euler's formula (5.1.1.1) by using a column length equal to twice the actual length *L* of the given column. We say that the *effective length* L_e of the column in the Fig.5.1.1*a* is equal to 2L and, substituting L_e = 2L in Euler's formula (5.1.1.1), we obtain Euler's formula (5.1.1.)

3rd buckling mode (end conditions)

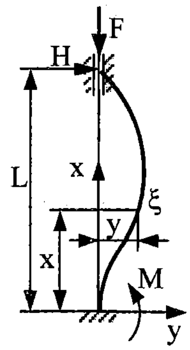


Fig.5.1.1.4

Based on Fig.5.1.1.3, it holds:

1) $M_\xi = Fy - H(L-x)$; 2) $y'' = -\frac{F}{EI}\left[y - \frac{H}{F}(L-x)\right] \Rightarrow$

$y'' + \alpha^2 y = \alpha^2 \frac{H}{F}(L-x)$. Its general solution is

$$y = A\cos\alpha x + B\sin\alpha x + \frac{H}{F}(L-x)$$

End (boundary) conditions:

1) $x=0\ ,\ y=0:\quad 0 = A\cos 0 + B\sin 0 + \frac{H}{F}(L-0) \quad\Rightarrow\quad A = -\frac{H}{F}L$

2) $x=0\ ,\ y'=0:\quad 0 = \alpha\left[-A\sin 0 + B\cos 0 - \frac{H}{\alpha F}\right] \quad\Rightarrow\quad B = \frac{H}{\alpha F}$

3) $x=L\ ,\ y=0:\quad 0 = \frac{H}{F}\left[\frac{1}{\alpha}\sin\alpha L - L\cos\alpha L\right] \Rightarrow \mathrm{tg}\alpha L = \alpha L$ (*indifferent equilibrium*).

The first root satisfying this transcendent equation is $\alpha L = 4.493$ and the critical load for the 3rd buckling mode is $F_{kr} = \frac{4.493^2 EI}{L^2} = \frac{20.16EI}{L^2}$, or $F_{crit} \cong \frac{2\pi^2 EI}{L^2} \cong \frac{\pi^2 EI_{min}}{(0.7L)^2}$ (5.1.1.3)

4th buckling mode (end conditions)

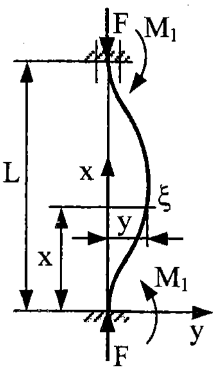


Fig.5.1.1.5

Based on Fig.5.1.1.4, it holds:

1) $M = Fy - M_1$; 2) $y'' = -\frac{F}{EI}\left[y - \frac{M_1}{F}\right] \Rightarrow$

$y'' + \alpha^2 y = \alpha^2 \frac{M_1}{F}$. Its general solution is

$$y = A\cos\alpha x + B\sin\alpha x + \frac{M_1}{F}$$

End (boundary) conditions:

1) $x=0\ ,\ y=0:\ 0 = A\cos 0 + B\sin 0 + \frac{M_1}{F} \quad\Rightarrow\quad A = -\frac{M_1}{F}$

2) $x=0\ ,\ y'=0: (y' = -A\alpha\sin\alpha x + B\alpha\cos\alpha x) \Rightarrow\ 0 = \alpha(-A\sin 0 + B\cos 0) \Rightarrow B = 0$

3) $x=L\ ,\ y=0:\ 0 = \frac{M_1}{F}(1-\cos\alpha L) \Rightarrow 1 = \cos\alpha L \Rightarrow \alpha L = k(2\pi)\quad;\quad k = 1,2,3...$

To obtain the minimum critical load we apply $k=1$: $L\sqrt{\frac{F}{EI}} = 2\pi \Rightarrow F_{crit} = \frac{4\pi^2 EI_{min}}{L^2}$ (5.1.1.3)

The region of applicability of *Euler's solution*:

We express so-called *buckling stress* by dividing the *critical loads* F_{crit} by the cross-sectional area *A,* and we obtain $\sigma_{crit} = F_{crit} / A$. In order to ensure the applicability of *Euler's solution* , we must stipulate that the *buckling stress* does not exceed the proportional point of the applied material, i.e., $\sigma_{crit} = F_{crit} / A \le \sigma_{prop}$. Introducing $I = A.i^2$, where *i* is the radius (of inertia) of the column profile, we can write successively

$$\sigma_{crit} = F_{crit} / A = \frac{n \cdot \pi^2 E I_{min}}{A \cdot L^2} = \frac{n \cdot \pi^2 E A \cdot i^2_{min}}{A \cdot L^2} = n \cdot \frac{\pi^2 E}{\left(\frac{L}{i_{min}}\right)^2} \Rightarrow \sigma_{crit} = n \cdot \frac{\pi^2 E}{\lambda^2} \le \sigma_{prop}$$

where λ is called "slenderness ". When considering the equality symbol, we obtain the so-called *limit slenderness* $\lambda_{lim} = \sqrt{n \frac{\pi^2 E}{\sigma_{prop}}}$ which divides columns into two categories:

1/ columns obeying *Euler's formulas*, i.e., $\lambda \ge \lambda_{lim}$ (elastic buckling, where $\sigma_{crit} \le \sigma_{prop}$)

2/ columns not obeying *Euler's formulas*, i.e. $\lambda < \lambda_{lim}$ (non-elastic buckling, $\sigma_{crit} > \sigma_{prop}$)

Buckling in the non-elastic region:

Note: There are a number of approaches to non-elastic buckling solutions:

a/ introduction of the *Equivalent elastic modulus,* tending to the extension of *Euler's formulas* also for the non-elastic region; b/ Application of *buckling coefficients;* c/ Application of *empirical formulas*

As the latter approach has greater practical significance, we will pay special attention to it:

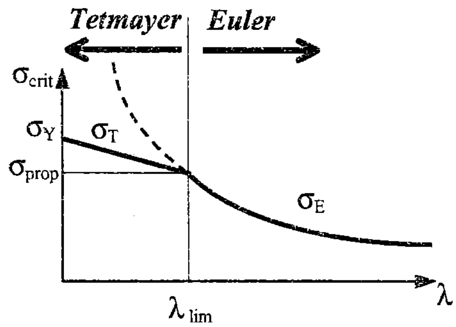
Its principle is based on experimentally obtained data to be used in the diagram $\{\sigma_{crit}, \lambda\}$.

The plots obtained have a shared point with *Euler's formulae* for σ_{crit} and they may obtain for instance the shapes of a parabola, a general curve, or a straight line.

In our country, we most frequently apply the *Tetmayer critical stress formulas:*

1/ for a *ductile* material $\quad \sigma_{crit} = \sigma_T = a - b\,(L/i) = a - b\,.\,\lambda$

2/ for a *brittle* material $\quad \sigma_{crit} = \sigma_T = a - b\,(L/i) + c\,(L/i)^2$



If we do not know the coefficients *a, b* for a *ductile* material we can fix the ends of the straight line $\sigma_T = f(\lambda)$ at points $[\lambda = 0;\ \sigma_T = \sigma_Y]$ and $[\lambda = \lambda_{lim};\ \sigma_T = \sigma_{prop}]$, thus obtaining the *Tetmayer critical stress*:

$$\sigma_{crit} = \sigma_T = \sigma_Y - \frac{\sigma_Y - \sigma_{prop}}{\lambda_{lim}} \lambda$$

5.2 Approximate solution methods

The above shown classical solution of column buckling can be applied only for simple stability problems (columns with a uniform cross-section and which are loaded at their ends). With columns that have a generally varying cross-section, or that are loaded along their length, the exact solution is no longer valid, and we must confine ourselves to:

a/ **energy methods** (by *Rayleigh)* and

b/ **the method of successive approximations** (by *Vianello)*

Note: A common feature of both methods is a ***supposition (estimation)*** *of the column deflection curve shape.* The column deflection shape is selected in such a way that it satisfies the column end conditions, corresponds with the course of the column stiffness along its length, and offers a simple solution.

5.2.1 Energy method

The basis of this method consists in comparing the column *strain energy increment ΔU* (produced when the strut is being bent) with the *external work increment ΔW* executed by the corresponding load F displacement ΔL: 1/ $\Delta U > \Delta W$, i.e., *stable equilibrium;* 2/ $\Delta U < \Delta W$, i.e., *instable equilibrium;* 3/ $\Delta U = \Delta W$; i.e., ***indifferent (neutral) equilibrium*.**

The energy method formula assessment will be presented for a pin-ended column, Fig.5.2.1.1:

a/ the column ***strain*** *energy increment ΔU*:

$$\Delta U = \frac{1}{2}\int_L \frac{M^2(x)}{EI}dx$$

Note: Moment $M(x)$ can be expressed:

α) in the shape $M(x) = F \cdot y$ holding for the 2nd buckling mode only ⇒

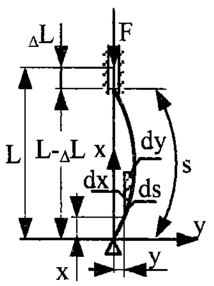
the increment of the strain energy holding for the 2nd buckling mode only is $\Delta U = \frac{1}{2}F^2\int_L \frac{y^2(x)}{EI}dx$

β) applying Bernoulli's equation for beam deflection $M^2(x) = (EI \cdot y'')^2$, we obtain the increment of the strain energy in a general form (applicable for all buckling modes): $\Delta U = \frac{1}{2}\cdot\int_L y''^2 EIdx$

b/ The load ***external*** *work increment ΔW:*

Neglecting the column length reduction (by compression) we can consider the column length to be unchanged after its bending $L \approx s$. For the column length element ds, it holds

$$ds = \sqrt{(dx^2 + dy^2)} = dx(1 + y'^2)^{\frac{1}{2}} \cong dx\left(1 + \frac{1}{2}y'^2\right)$$



5.2.1.1

(when, for a small column deflection, the development in Taylor's series applies)

Beginning from the infinitesimal load (F) displacement (related to element ds)

$$d\Delta L = ds - dx = dx\left(1 + \frac{1}{2}y'^2\right) - dx = \frac{1}{2}y'^2 dx$$

we obtain load (F) displacement ΔL by integration: $\Delta L = \frac{1}{2}\int_L y'^2 dx$

The external work executed by the force *F* (which remains constant when producing the work *ΔW* along *ΔL*) is $\Delta W = F \cdot \Delta L = \frac{F}{2} \int_L y'^2 dx$

c/ The critical force is assessed from the equality *ΔW* = *ΔU* (obtained in two forms):

ad α) $\frac{1}{2} F^2 \int_L \frac{y^2(x)}{EI} dx = \frac{F}{2} \int_L y'^2 dx \Rightarrow F_{crit} = \frac{\int_L y'^2 dx}{\int_L \frac{y^2(x)}{EI} dx}$ (2nd buckling mode only) (5.2.1.1)

ad β) $\frac{1}{2} \int_L y''^2 EI \, dx = \frac{F}{2} \int_L y'^2 dx \Rightarrow F_{crit} = \frac{\int_L y''^2 EI \, dx}{\int_L y'^2 dx}$ (all buckling modes) (5.2.1.2)

Note 1: The solution obtained from formula (5.2.1.1) can be satisfactory even for a very *approximate* (non-accurate) estimation of the column deflection curve function (because in this case the second derivation, which is very difficult to estimate, plays no role). While the solution obtained from formula (5.2.1.2) can be satisfactory only when applying the *exact* deflection curve (if we know this from another solution), or when expressing it in the form of a series.

Note 2: The energy method is based on the *theorem of minimum potential energy* $\pi = W - U$: $\delta\pi = \delta W - \delta U = 0 \Rightarrow \delta W = \delta U$, which holds exactly only when the deflection curve is accurately found (estimated). It can be proved that $(\delta)^2\pi > 0 \Rightarrow$ *minimum.* Because we rarely find the exact deflection curve, we must apply the symbol ≤ , instead of =, in the two formulas (5.2.1.1) and (5.2.1.2)

5.2.2 Method of successive approximations (by Vianello)

When applying this method, we start from a selected (estimated) shape of the column deflection curve $y_o(x)$ (this may in fact be a graft only - corresponding to the column dimensions but not necessary obeying its end conditions). The column is to be loaded with a load F_o (assumed to be the respective critical load), because a column is bent only when in *indifferent* (neutral) *equilibrium.*

The moment at a column arbitrary point is then $M(x) = F_o \cdot y_o(x)$ and the differential equation of the deflection curve is expressed $y_1''(x) = -\frac{M(x)}{EI} = -\frac{F_o \cdot y_o(x)}{EI}$, where $y_1(x)$ is a new deflection curve which has to obey the column end conditions. This curve $y_1(x)$ can then serve as a newly starting deflection curve from which we obtain $y_2(x)$. This will fit better to the exact curve. After *n-1*-iteration cycles we can write $y_n''(x) = -\frac{F_o \cdot y_{n-1}(x)}{EI}$. This newly obtained curve is compared with the previous curve. The procedure is iterated as long as $y_n(x) : y_{n-1}(x) = K \pm \varepsilon$, where K is a constant and ε a selected admissible deviation of two successive deflection curves. In such a way, we obtain (with the required accuracy) the deflection curve shape, which approximates to the *eigen function* of the critical load. In this case, it holds for $y_n(x)$ that:

i) $EI\, y_n''(x) = -F_o\, y_{n-1}(x)$;

ii) and $EI\, y_n''(x) = -F_{crit}\, y_n(x)$ from which it follows: $F_{crit} = F_o \cdot y_{n-1}(x) / y_n(x)$

Note: As the procedure exertion increases with the increasing number of iterations, we endeavour for the most effective exploitation of the two lowest possible successive approximations, for instance the selected $y_o(x)$ and computed $y_1(x)$ or, in an extreme case, from the couple of deflection curves $y_1(x)$ and $y_2(x)$ resulting from the next approximation. Since, in such cases, the single ratios $y_0(x) / y_1(x)$ can differ somewhat for different *x*, we involve more deflection values $(y_n(x))$ in the critical load computation

$$F_{crit} = F_1 = F_0 \frac{\Sigma y_{0i}}{\Sigma y_{1i}}, \quad \text{or} \qquad F_{crit} = F_1 = F_0 \frac{\int y_0(x)dx}{\int y_1(x)dx} = F_0 \frac{A_0}{A_1}$$

5.3 Combined stress: buckling & bending

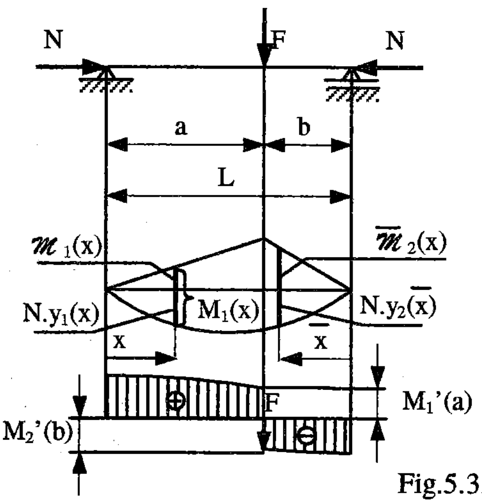


Fig.5.3.1

In practice there are often cases where a beam, while carrying transversal forces, is simultaneously subjected to axial loading. Such a case is especially unfavourable when the axial load is compressive. Although this ***is not a stability problem***, we currently denote it as ***buckling & bending** combined stress.* A transversal loading (being supposed to act in one of the principal planes of the beam) produces a basic bending moment $\mathcal{M}$ and brings about the beam bending with deflection y_b in this plane. Owing to the deflection, the axial force *N* acts eccentrically in the beam individual cross-sections, producing an additional moment $M_N = N \cdot y_b$ together with a gradually increasing deflection y_N added to the basic deflection y_b so that the final deflection magnitude is

$$y_{(x)} = y_b + y_N \tag{5.3.1}$$

A resulting moment in an arbitrary cross-section *x* is then

$$M_{(x)} = \mathcal{M}_{(x)} + Ny_{(x)} \tag{5.3.2}$$

This type of loading in fact represents a *combination of* ***compression & bending*** that **means not a *stability* problem but a *strength* problem.** So that it must satisfy the basic *strength condition*

$$|\sigma|_{max} = \frac{M_{max}}{Z_b} + \frac{N}{A} \le \sigma_{all} \tag{5.3.3}$$

Nevertheless, we must, at the same time, ***check a corresponding stability condition*** of the beam, because its carrying capacity failure might set in the direction of *minimum stiffness* of the beam, which would be caused by *buckling* .

An assessment of the maximum moment M_{max} of buckling & bending combination to be substituted into the *strength criterion* (Eq.5.3.3):

1) *Solution based on the deflection curve y(x)*: When substituting Eq.(5.3.2) into *Bernoulli's equation* and introducing $\frac{N}{EI} = \alpha^2$, we have successively: $y'' + \alpha^2 y = -\frac{\alpha^2}{N}\mathcal{M}_{(x)}$ *),

the general solution of which is: $y_{(x)} = A\cos\alpha x + B\sin\alpha x + y_p$

This represents a resulting deflection curve, and its substitution in Eq.(5.3.2) yields M_{max}.

Note: *A, B* are integration constants to be assessed from the boundary conditions: for $x = 0 \rightarrow y(0) = 0$; for $x = L \rightarrow y(L) = 0$; or, if need be, from the *conditions of beam deflection curve compatibility*, i.e., the *equality* of the *deflection magnitudes* and the *deflection slopes* from the two sides of the *beam cross-section under consideration*; y_P is a particular integral of the non-homogeneous differential equation.

2) *Direct moment solution M(x):* Starting from the double derivation of Eq.(5.3.2) we have $M''(x) = \mathcal{M}''(x) + N \cdot y''(x)$, where we substitute *Bernoulli's equation* and thus obtain

$M''(x) + \alpha^2 M(x) = \mathcal{M}''(x)$ **). The general solution is $M_{(x)} = A\cos\alpha x + B\sin\alpha x + M_p$

Note: The integration constants *A, B* are similarly obtained from the respective boundary conditions, and M_P is a particular integral. By finding an extreme, we obtain M_{max}.

Example: *Assess M_{max} produced in the column shown in Fig.5.3.1.*

Solution: Transversal force *F* produces bending moments in two spans (*a, b*):

$\mathcal{M}_1(x) = F\frac{b}{L}x$, $\mathcal{M}_2(\bar{x}) = F\frac{a}{L}\bar{x}$ (i.e., $\mathcal{M}_1''(x) = 0$; $\mathcal{M}_2''(\bar{x}) = 0$), thus Eq.**) is *homogeneous.*

Span $0 \le x \le a$: $M_1'' + \alpha^2 M_1 = 0$ (from left)	*Span* $0 \le \bar{x} \le b$: $M_2'' + \alpha^2 M_2 = 0$ (from right)
$M_1(x) = A_1\cos\alpha x + B_1\sin\alpha x$	$M_2(\bar{x}) = A_2\cos\alpha\bar{x} + B_2\sin\alpha\bar{x}$
$V_1 = M_1' = -\alpha A_1\sin\alpha x + \alpha B_1\cos\alpha x$	$V_2 = -M_2' = \alpha A_2\sin\alpha\bar{x} - \alpha B_2\cos\alpha\bar{x}$

BC: 1) $x = 0$: $M_1(0) = 0 \Rightarrow A_1 = 0$, thus $M_1(x) = B_1\sin\alpha x$; 2) $x = 0$: $M_2(0) = 0 \Rightarrow A_2 = 0$

3) $x = a$ and $\bar{x} = b$: $M_1(a) = M_2(b) \Rightarrow B_1 = \frac{F\sin\alpha b}{\alpha\sin\alpha L}$, thus $M_1(x) = \frac{F\sin\alpha b}{\alpha\sin\alpha L}\sin\alpha x$

4) $x = a$ and $\bar{x} = b$: $M_1'(a) - [-M_2'(b)] = F \Rightarrow B_2 = \frac{F\sin\alpha a}{\alpha\sin\alpha L}$, $M_2(\bar{x}) = \frac{F\sin\alpha a}{\alpha\sin\alpha L}\sin\alpha\bar{x}$

Note: For a case, where the transversal load *F* is acting in the beam centre ($a = b = L/2$), the extreme location is in $x_m = \frac{L}{2}$ and the maximum moment obtains the value as follows $M_{max} = \frac{FL}{4}\left(\frac{tg\frac{\alpha L}{2}}{\frac{\alpha L}{2}}\right) = \mathcal{M}_{max} \cdot \varphi$

6. Plasticity

6.1 Introduction

So far we have dealt with elastic behaviour of structures, i.e., structures that have been stressed within the range of Hooke's Law, i.e., to the proportional limit of the material. Now, we extend structure stressing ***beyond the proportional limit***. As material stress-strain curves have complicated shapes after exceeding this boundary, and thus a computation of the structure stress distributions can hardly be carried out by an analytical approach, ***simplified** material models* are introduced. When applying a *tensile test* on a specimen of a **ductile material** (see [1], Chap.2, Fig.2.6.1.1), we can observe that, after a critical value σ_Y (point Y) of the stress has been reached, the specimen undergoes a large deformation with a relatively small increase in the applied load. (This deformation is caused by slippage of the material along oblique surfaces and is due, therefore, primarily to shearing stresses.) When comparing $\varepsilon_Y \approx$ **0.1%** (the strain corresponding to point Y) with $\varepsilon \approx$ **1.6%**, which occurs at the end of the slippage (where material hardening begins), and taking into account that, though we will deal with the plastic behaviour of the material, only a very small structure deformation will be considered, a very realistic model of material is that of *ideal plasticity* (material hardening being neglected), the so-called ***elastic-ideally plastic model** of material*, Fig.6.1.1*a*. If more complicated structures are dealt with (e.g., plates or shells) we have an even simpler model, the so-called ***rigid-ideally plastic model** of material* (where elastic strain is neglected completely), Fig.6.1.1*b*.

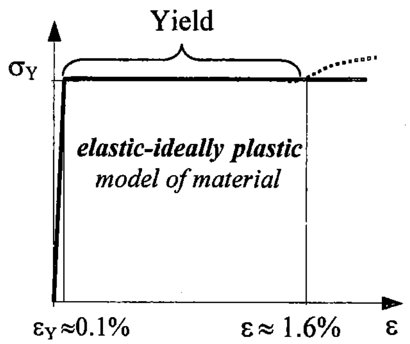


Fig.6.1.1*a*

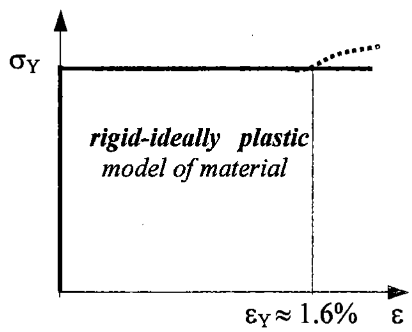


Fig.6.1.1*b*

We will be concerned with basic types of loading: i) *tension and compression of rods*; ii) *torsion of various profiles*; iii) *bending of beams*; iv) *pressurizing of thick cylinders*. The concept of the ***plastic limit load*** of structures will be explained when dealing with tension and compression of rods and it will then be applied to other types of loading.

6.1.1 Definition of the limit carrying capacity (plastic limit state) of a structure

When loading a structure from zero load parameter ($\boldsymbol{p} = \boldsymbol{0}$) it will behave elastically up to a certain load parameter p_{el} called the *elastic limit load*. After passing this limit, a plastic flow will occur in some parts of the structure where the stress states reach the *yield strength* of the structure material σ_Y. The higher the load the more extensive the plastic regions will be until: i)either *all the structure stress state* obeys the *yield condition* of the structure material (i.e., the structure will undergo plastic flow), or ii), the *remaining elastic* parts are not able to *bear the structure in equilibrium*, and the structure will become a *limit mechanism* no longer able to bear its operational load. *Such a stress state is called the* ***limit state*** *of the structure,* and *its corresponding load parameter* p_{lim} *is called the* ***limit load carrying capacity*** *(plastic limit,* or *collapse load) of the structure.*

6.2 Tension and compression of rods beyond the validity of Hooke's Law

A statically indeterminate *pin-connected framework*, Fig.6.2.1, is examined while being loaded with load *F* gradually increasing from *zero* up to *a limit load*: $0 \le F \le F_{lim}$. This procedure will contain *2 stages*:

1) The pin-connected framework behaviour within the range of the validity of Hooke's law $(0 \le F \le_{el})$

The problem being *statically indeterminate to the first degree* (1° SI), we apply ***one compatibility*** *equation* together with ***two*** *equilibrium equations,* which yields (cf. [1], Chap.3, Sec.3.5, pp 44-45):

$N_1 = N_2 = \dfrac{F\cos^2\alpha}{1+2\cos^3\alpha}$; $N_3 = \dfrac{F}{1+2\cos^3\alpha}$. Maximum stress

$\sigma_{max} = \sigma_3 = \dfrac{N_3}{A} = \sigma_Y$ defines the *elastic limit load* of the framework

$F_{el} = \sigma_Y \cdot A(1+2\cos^3\alpha)$ *). When divided by a factor of safety it

yields *allowable load* $F'_{all} = \dfrac{F_{el}}{k} = \dfrac{\sigma_Y}{k} \cdot A(1+2\cos^3\alpha)$ (6.2.1)

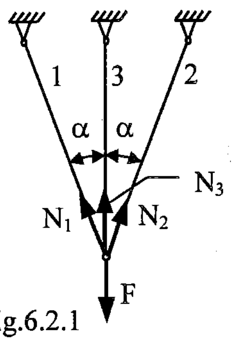


Fig.6.2.1

2) Elastic-plastic behaviour of the pin-connected framework, i.e., $F_{el} < F \le F_{lim}$

Since the stress in rod *3* has reached yield strength σ_Y, it cannot increase (***elastic-ideally plastic*** *model of material,* Fig.6.1.1*a*). The framework, being changed into an *SD* problem (Fig.6.2.2), can be solved exclusively by static means, i.e., the force equilibrium in the vertical direction: $(N_1+N_2)\cos\alpha + A\sigma_Y - F = 0 \Rightarrow N_1 = N_2 = N = \dfrac{F - A\sigma_Y}{2\cos\alpha}$

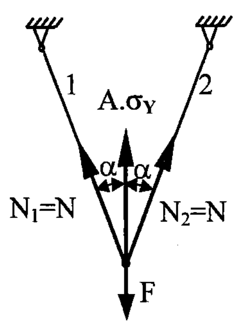


Fig.6.2.2

When yield strength σ_Y is also reached in the two lateral rods: $\sigma_1 = \sigma_2 = \dfrac{N}{A} = \dfrac{F_{lim} - A\sigma_Y}{A \cdot 2\cos\alpha} = \sigma_Y$, the framework becomes a ***collapse mechanism*** and the load: $F_{lim} = A\sigma_Y(1+2\cos\alpha)$ (6.2.2)

(which caused this collapse of the framework) is called the *limit (plastic) load.*

When divided by a safety factor, this yields another *allowable load* $F''_{all} = \dfrac{F_{el}}{k} = \dfrac{\sigma_Y}{k} \cdot A(1+2\cos\alpha)$

Note: The ratio $\beta = \dfrac{F_{lim}}{F_{el}} = \dfrac{F''_{all}}{F'_{all}} = \dfrac{1+2\cos\alpha}{1+2\cos^3\alpha}$ means a *gain* of the limit plastic analysis. ($\alpha = 60°$: $\beta = \dfrac{8}{5}$).

Instant assessment of the limit load F_{lim}:

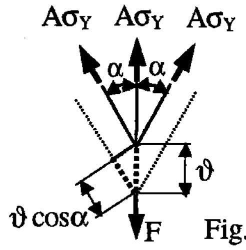


Fig.6.2.3

Based on the collapse mechanism (Fig.6.2.3) we can assess the limit load by means of the ***method of virtual work***: $D_e = D_i$ (6.2.3)
(D_e...dissipation of external energy; D_i...dissipation of internal energy)

$$\Rightarrow F_{lim} \cdot \vartheta = \sigma_Y \cdot A \cdot \vartheta + 2 \cdot \sigma_Y \cdot A \cdot \vartheta \cdot \cos\alpha \Rightarrow F_{lim} = A\sigma_Y (1 + 2\cos\alpha)$$

Note: Advantages of (plastic) limit analysis (deduced from the foregoing results):

1/ The (plastic) limit analysis of a structure is often *simpler* than the elastic analysis.
2/ A better overview of the behaviour of a structure is achieved when studying its *limit carrying capacity* (plastic limit state), which means that we can design a *safer structure in a logical sense* (although formally having a lower coefficient of safety than with the elastic solution).
3/ Structures designed by (plastic) limit analysis have a *lower cost.*

6.2.1 Residual stress and strain

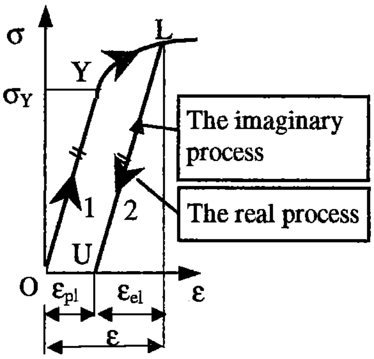


Fig.6.2.1.1

The idea of residual stress can be readily obtained by understanding the phenomenon of residual strain. From Fig.6.2.1.1 we can deduce:
Curve 1...the *loading process* consists of an *elastic part (O→ Y)* and a *plastic part (Y→ L)*
Curve 2...the *unloading process* (*L→U*) passes *parallel* with the *elastic part (O→ Y)* of the loading process **1** ⇒
The **unloading process is always *elastic*.**

But as we do not know how to compute the unloading process we will compute *an imaginary loading process U→L* which we subtract from the loaded state represented by point *L*. Thus

$$\varepsilon_{res} = \varepsilon_{pl} = \varepsilon - \varepsilon_{el}\,.$$

In the same manner we define the residual stress: ***Residual stress*** $\boldsymbol{\sigma_{res}}$ is obtained when we subtract from the stress $\boldsymbol{\sigma} = \mathbf{f_1(F_w)}$, i.e., *O→L* (produced by a working load $F_{el} < F_w \le F_{lim}$), the imaginary stress $\boldsymbol{\sigma_f} = \mathbf{f_2(F_w)}$, i.e., *U→L,* which we obtain by loading the structure with the same working load F_w but when we consider that it always behaves elastically:

$$\sigma_{res} = \sigma_{pl} = \sigma - \sigma_f \qquad (6.2.3.1)$$

6.3 Plastic torsion

6.3.1 Circular profiles

Based on the ***elastic-ideally plastic model*** *of material* and Fig.6.3.1.1, we can express:

1) *Elastic limit torque*: $T_{el} = \frac{\pi}{16} d^3 \tau_Y = \frac{\pi}{2} r^3 \tau_Y$;

2) *Elastic-plastic torque*:

$$T_{elpl} = \frac{\pi}{2}\rho^3\tau_Y + \int_\rho^r 2\pi\rho\rho\tau_Y d\rho = \frac{2}{3}\pi\tau_Y\left(r^3 - \frac{\rho^3}{4}\right) = \frac{\pi}{2} r^3 \tau_Y \left[\frac{4}{3} - \frac{1}{3}\left(\frac{\rho}{r}\right)^3\right];$$

3) *Plastic limit torque ($\rho \to 0$)*: $T_{lim} = T_{el} \cdot \frac{4}{3} = \frac{2\pi}{3} r^3 \tau_Y$

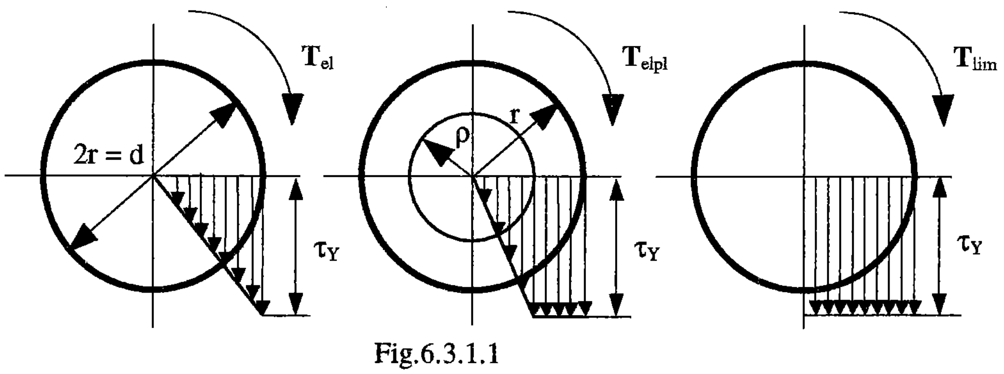


Fig.6.3.1.1

6.3.2 Non-circular profiles

Utilizing the stress-cap characteristics: 2nd (the τ magnitude is given by the stress-cap gradient **tgα**), and 3rd ($T = 2V$, where V is the stress-cap volume), and the ***elastic-ideally plastic model*** *of material* to an assessment of the torque borne by a rectangle profile (Fig.6.3.2.1) we have

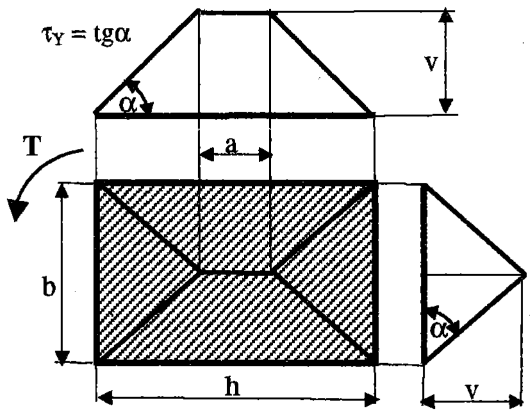


Fig.6.3.2.1

$$T = 2V = \frac{b \cdot v}{2} a + \frac{b^2 \cdot v}{3} = \tau_y \cdot \left(\frac{b^2 \cdot h}{4} - \frac{b^3}{12} \right)$$

$$\text{where } v = \frac{b}{2} \cdot tg\alpha = \frac{b}{2} \cdot \tau_y;\ a = h - b$$

6.4 Plastic bending

Fig.6.4.1 summarises all the problems we can encounter with beams undergoing bending beyond the validity of Hooke's law: I) The profile stress distributions at the central section ($x = L/2$), are maximally stressed, which enables us to assess various types of moments (M_{el}, M_{elpl}, M_{lim}); II) Moment distributions (corresponding to the profile stress distributions) leading to the definition of the *beam collapse mechanism* and the corresponding (*plastic*) *limit load* (F_{lim}); III) Ranges of plasticity in the beam center corresponding to the load intensity.

I) Types of bending moments based on the profile stress distributions at the central section ($x = L/2$):

1) *Elastic limit (bending) moment:* $M_{el} = Z_b \sigma_Y = \frac{1}{6} b h^2 \sigma_Y$ (6.4.1)

2) *Elastic-plastic moment*: $M_{elpl} = \frac{1}{6} b a^2 \sigma_Y + 2\int_{\frac{a}{2}}^{\frac{h}{2}} \sigma_Y \cdot b \cdot y \cdot dy = \frac{h^2 b}{4} \sigma_Y \left[1 - \frac{4}{3} \left(\frac{a}{2h} \right)^2 \right]$ (6.4.2)

3) *Plastic limit moment* (for $a \to 0$ we obtain an ideal *limit plastic moment* which appears in the beam as an ideal *plastic hinge*): $M_{pl} \cong \frac{1}{4} b h^2 \sigma_Y$ (6.4.3)

II) Assessment of the *limit load* of the beam:

We can apply two methods:

1) *Comparison of internal and external moments,* i.e., $\frac{F_{lim} L}{4} = M_{pl} \quad \Rightarrow F_{lim} = \frac{4M_{pl}}{L}$

2) *The method of virtual works:* $D_e = D_i$

Consulting Fig.6.4.1 we have: $D_e = F \cdot u = F \cdot \vartheta \cdot \frac{L}{2}$, and $D_i = M_{pl} \cdot 2\vartheta \Rightarrow \quad F_{lim} = \frac{4M_{pl}}{L}$

(D_e ...dissipation of external energy; D_i ...dissipation of internal energy)

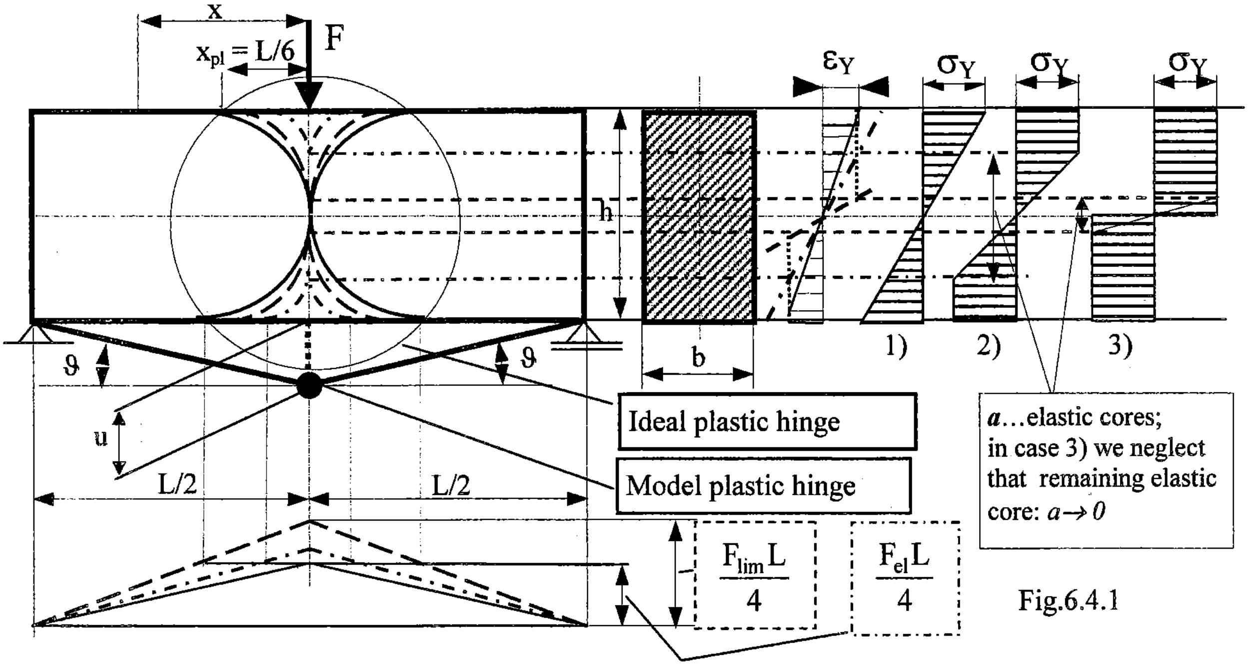


Fig.6.4.1

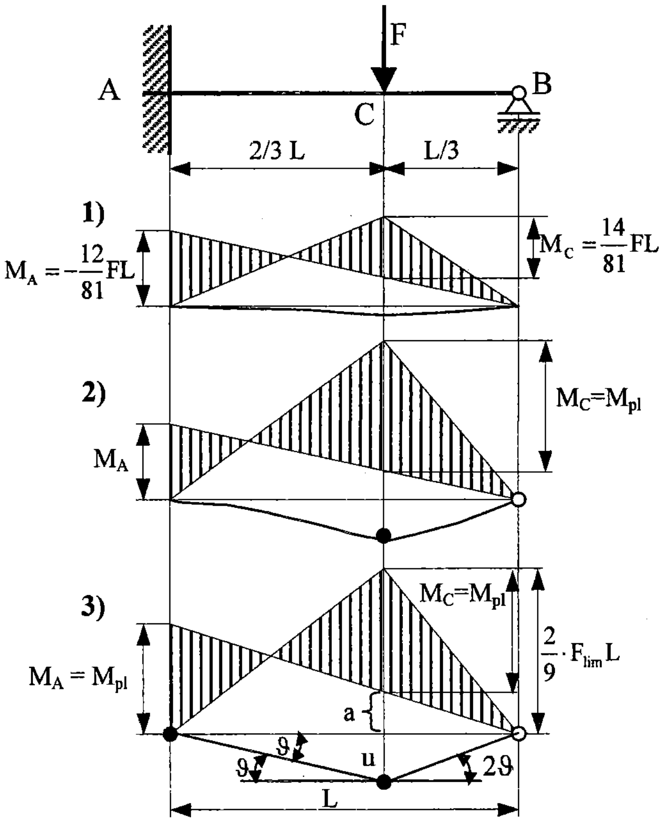
III) Range of plasticity in the beam centre

The range of plasticity aroused in a loaded beam can be obtained by comparing the *external moment distribution M(x)* with *the internal elastic-plastic moment* $M_{elpl}(y)$ (elastic core $y = a/2$). When dealing with the beam in Fig.6.4.1, we have a parabola:

$M(x) = M_{elpl}(y) \Rightarrow \frac{F}{2}\left(\frac{L}{2} - x\right) = M_{pl}\left[1 - \frac{4}{3}\left(\frac{y}{h}\right)^2\right]$. To obtain the *plastic range border* at the *plastic collapse of the beam,* we substitute $F = F_{lim} = \frac{4M_{pl}}{L}$ and have $\left(\frac{y}{h}\right)^2 = \frac{3}{2} \cdot \frac{x}{L}$ *)

The greatest distance x_{pl} of the material plastic state measured from the beam centre is at the upper and lower faces of the beam, i.e., when we substitute $y = h/2$ in *): $x_{pl} = \frac{L}{6}$.

Example: A statically indeterminate cantilever is subjected to concentrated load *F*. Analyse the problem when load *F* gradually increases up to the limit carrying capacity F_{lim} of the cantilever, Fig.6.4.2.



● Plastic hinge; ○ Designed hinge; Fig.6.4.2

1) $0 \le F \le F_{el}$ The problem is 1° SI.

Comp. eq.: $\varphi_A = 0 \Rightarrow M_A = -\frac{12}{81}FL$

$$M_{max} = M_C = \frac{14}{81}FL = M_{pl} \Rightarrow$$

$$\Rightarrow F_{el} = \frac{81}{14}\cdot\frac{M_{pl}}{L}$$

2) $F_{el} \le F \le F_{lim} \Rightarrow$ SD problem

The moment eq. about *C*: $R_B = \frac{3M_{pl}}{L} \Rightarrow$

$$|M_A| = \left|R_B L - \frac{2}{3}FL\right| \ge \frac{6}{7}M_{pl}$$

will grow up to $|M_A| = M_{pl} \Rightarrow$

$$|M_A| = -\left(3M_{pl} - \frac{2}{3}FL\right) = M_{pl} \Rightarrow$$

$$F_{lim} = \frac{6M_{pl}}{L}$$

A **direct** assessment of the *plastic limit load* (based on a kinematically admissible collapse mechanism):

1) *The comparison of internal and external moments,*

i.e., $\frac{2}{9}F_{lim}L = M_C + a = M_{pl} + \frac{M_{pl}}{3} \quad \Rightarrow \quad F_{lim} = \frac{6M_{pl}}{L}$

2) *The method of virtual works:* $D_e = D_i \Rightarrow$

$$F_{lim}\cdot u = M_{pl}\cdot(4\vartheta) \Rightarrow F_{lim}\cdot\left(2\vartheta\cdot\frac{L}{3}\right) = M_{pl}\cdot(4\vartheta) \quad \Rightarrow \quad F_{lim} = \frac{6M_{pl}}{L}$$

6.5 Plastic behaviour of thick cylinders under inner overpressure

We commence with the equilibrium equation $\sigma_r - \sigma_t + x\cdot\frac{d\sigma_r}{dx} = 0$ (6.5.1)

which does not depend on the material properties. Now, considering that the pressure vessel is made of a ductile material, we are to assess a relevant plastic criterion, which could be *Tresca's yield criterion*

$$\sigma_{eq} = \sigma_{max} - \sigma_{min} = \sigma_y \tag{6.5.2}$$

Recalling the **elastic** stress distribution for this case we conclude that the maximum stress is σ_t, while the minimum stress is σ_r, at all the cylinder points described by a general radius *x*, and it is evident that, although the plastic stress distribution will look different, this succession will not change. Applying this conclusion, Eq.(6.5.2) will have the shape $\sigma_t - \sigma_r = \sigma_y$ (6.5.3)

Substituting Eq.(6.5.3) into Eq.(6.5.1) we obtain a simple differential equation

$$\frac{d\sigma_r}{dx}x = \sigma_y \qquad (6.5.4)$$

the solution of which is

$$\sigma_r = \sigma_y \cdot \ln x + C \qquad (6.5.5)$$

Although it has <u>only **one** *integration constant*, **two** *boundary conditions* are available</u>:

1/ for $x = r_1$, it holds $\sigma_{r1} = -p_1 = \sigma_y \cdot \ln r_1 + C$, which, denoting the **starting** *plastic stress state*, will serve for determining the integration constant, $C = -p_1 - \sigma_y \cdot \ln r_1$, resulting in the *plastic* ***radius** stress distribution*

$$\sigma_r = -p_1 + \sigma_y \cdot \ln\frac{x}{r_1} \qquad (6.5.6)$$

2/ for $x = r_2$, it holds $\sigma_{r2} = -p_2 = -p_1 + \sigma_y \cdot \ln\frac{r_2}{r_1}$, which (denoting that the *plastic stress state* is distributed right to the outer radius of the cylinder, i.e., the **whole cylinder undergoes** *plastic flow*) will serve for obtaining the <u>cylinder **limit** plastic overpressure</u>:

$$(p_1 - p_2)_{lim} = \sigma_y \cdot \ln\frac{r_2}{r_1} \qquad (6.5.7)$$

Finally, from the *Tresca's yield criterion* and the *plastic radial stress distribution*, see Eq.(6.5.3) and (6.5.6), respectively, we express the *plastic **tangential** stress distribution*

$$\sigma_t = -p_1 + \sigma_y \cdot \left(1 + \ln\frac{x}{r_1}\right) \qquad (6.5.8)$$

The plastic tangential and radial stress distribution for a cylinder under inner overpressure is shown in Fig.6.5.1

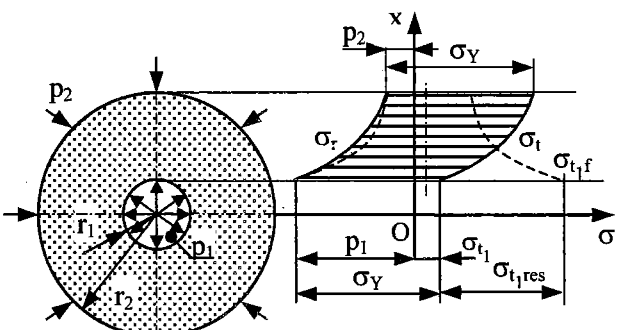


Fig.6.5.1

Note: The residual stress on the inner face after unloading from the plastic limit overpressure $(p_1)_{lim}$, for $p_2 = 0$:

$$\sigma_{t_1 res} = \sigma_{t_1} - \sigma_{t_1 f} = (-p_1 + \sigma_Y) - \left(2\frac{p_1 r_1^2}{r_2^2 - r_1^2} + p_1\right) =$$

$$= \sigma_Y - 2p_1\frac{r_2^2}{r_2^2 - r_1^2} = -\sigma_Y\left[2\ln\frac{r_2}{r_1}\cdot\frac{r_2^2}{r_2^2 - r_1^2} - 1\right]$$

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